1. (15) Define the following terms. With each term include an example of what you are defining. No example, no credit.

(a) Linear transformation from a vector space $U$ into a vector space $V$.

A linear transformation, $L$, from $U$ to $V$ is a function with domain $U$ and co-domain $V$ such that

$$L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$$
$$L(\alpha \vec{x}) = \alpha L(\vec{x})$$

An example of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ is

$$L(x_1, x_2) = (0, 0).$$

(b) Length of a vector $\vec{x}$ in $\mathbb{R}^5$.

If $\vec{x} = (x_1, x_2, x_3, x_4, x_5)$, then $||\vec{x}|| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2}$. A specific example is

$$||(1, 1, 0, 0, 0)|| = \sqrt{2}.$$

(c) Vector projection of $\vec{x}$ onto $\vec{y}$, where $\vec{x}$ and $\vec{y}$ are two vectors in $\mathbb{R}^n$.

The projection of $\vec{x}$ onto $\vec{y}$ equals $\frac{\vec{x} \cdot \vec{y}}{||\vec{y}||^2} \vec{y}$. An example in $\mathbb{R}^2$ is

$$\text{Proj}_{(1,0)} (3, 4) = \frac{3}{1} (1, 0) = (3, 0).$$

2. (10) Let $\vec{x} = (1, 2, -5, 1)$ and $\vec{y} = (3, -1, 1, 2)$.

(a) What is the length of $\vec{x}$?

The length of $\vec{x}$ equals

$$\sqrt{1 + 4 + 25 + 1} = \sqrt{31}.$$ 

(b) Are the vectors $\vec{x}$ and $\vec{y}$ perpendicular?

The dot product of $\vec{x}$ and $\vec{y}$ equals $3 - 2 - 5 + 2 = -2$. Since the dot product is not equal to zero, the vectors are not perpendicular.
3. (25) Let $V = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_4 = 0\}$.

(a) Find a basis of $V$.

The coefficient matrix of the system of equations $x_1 + x_2 + x_4 = 0$ is

$$
\begin{bmatrix}
1 & 1 & 0 & 1
\end{bmatrix}
$$

The free variables are $x_2, x_3,$ and $x_4$. Thus, a basis of $V$ is

$$\{(-1, 1, 0, 0), (0, 0, 1, 0), (-1, 0, 0, 1)\}.$$

(b) Find an orthonormal basis of $V$.

Call the vectors in the basis from part a. $\vec{v}_1, \vec{v}_2,$ and $\vec{v}_3$ respectively. Then an orthonormal basis of $V$ is

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}} (-1, 1, 0, 0)$$

$$\vec{u}_2 = \frac{\vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1}{\|\vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1\|} = \frac{(0, 0, 1, 0) - 0 \vec{u}_1}{\|\vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1\|} = (0, 0, 1, 0)$$

$$\vec{u}_3 = \frac{\vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2}{\|\vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2\|}$$

$$= \frac{\vec{v}_3 - \frac{1}{2} \vec{u}_1 - 0 \vec{u}_2}{\|\vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2\|} = \frac{1}{\sqrt{6}} (-1, -1, 0, 2)$$

(c) Find the projection of $\vec{x} = (1, 1, 1, 1)$ onto $V$.

The projection of $\vec{x}$ onto $V$ equals

$$\text{Proj}_V \vec{x} = (\vec{x} \cdot \vec{u}_1) \vec{u}_1 + (\vec{x} \cdot \vec{u}_2) \vec{u}_2 + (\vec{x} \cdot \vec{u}_3) \vec{u}_3$$

$$= 0 \vec{u}_1 + \vec{u}_2 + 0 \vec{u}_3 = \vec{u}_2$$

$$= (0, 0, 1, 0)$$

(d) Find the distance from $\vec{x} = (1, 1, 1, 1)$ to $V$.

The distance from $\vec{x}$ to $V$ equals

$$\|\vec{x} - \text{Proj}_V \vec{x}\| = \|(1, 1, 1, 1) - (0, 0, 1, 0)\|$$

$$= \|(1, 1, 0, 1)\|$$

$$= \sqrt{3}.$$
4. (20) Let \( L : P_3 \rightarrow M_{2,2} \) be defined by

\[
L(p) = \begin{bmatrix} p(0) & p(2) \\ p(0) + p(2) & \int_0^2 p(x) \, dx \end{bmatrix}.
\]

(a) Compute \( L(1) \).

\[
L(1) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.
\]

(b) Find the matrix representation of \( L \) with respect to the standard bases of \( P_3 \) and \( M_{2,2} \). These bases are \( \{1, t, t^2\} \) and \( \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\} \) respectively.

\[
L(1) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad L(t) = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}, \quad L(t^2) = \begin{bmatrix} 0 & 4 \\ 4 & 8/3 \end{bmatrix}.
\]

The coordinates of these vectors with respect to the standard basis in \( M_{2,2} \) are

\[
\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 4 \\ 4 \\ 8/3 \end{bmatrix}
\]

Thus, the matrix representation of \( L \) with respect to the standard bases is

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 2 & 2 & 4 \\ 2 & 2 & 8/3 \end{bmatrix}.
\]
5. (15) Find an equation for the straight line which best fits the set of points \{(-2, 2), (-1, 0), (1, 1), (3, 2)\}.

If \( y = mx + b \) is the equation of the line, the coefficients \( m \) and \( b \) should satisfy the system of equations

\[
\begin{align*}
-2m + b &= 2 \\
-m + b &= 0 \\
m + b &= 1 \\
3m + b &= 2
\end{align*}
\]

Writing this as a matrix equation we have \( A\vec{x} = \vec{y} \) where

\[
A = \begin{bmatrix}
-2 & 1 \\
-1 & 1 \\
1 & 1 \\
3 & 1
\end{bmatrix}, \quad \text{and} \quad \vec{y} = \begin{bmatrix}
2 \\
o \\
1 \\
2
\end{bmatrix}.
\]

This system does not have a solution, so the normal equations are used to find the least squares solution

\[
\vec{x} = (A^T A)^{-1} A^T \vec{y}
\]

\[
= \begin{bmatrix}
15 & 1 \\
1 & 4
\end{bmatrix}^{-1} \begin{bmatrix}
-2 & -1 & 1 & 3 \\
1 & 1 & 1 & 1
\end{bmatrix} \begin{bmatrix}
2 \\
o \\
1 \\
2
\end{bmatrix}
\]

\[
= \begin{bmatrix}
4/59 & -1/59 \\
-1/59 & 15/59
\end{bmatrix} \begin{bmatrix}
3 \\
5
\end{bmatrix} = \begin{bmatrix}
7/59 \\
72/59
\end{bmatrix}.
\]

Thus, the best straight line fit to the data is

\[
y = \frac{7}{59}x + \frac{72}{59} \approx 0.119x + 1.22.
\]
6. (15) Suppose that $L$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$. Let $\vec{v}_1 = (1, -3)$ and $\vec{v}_2 = (3, 1)$. Suppose that $L(\vec{v}_1) = \vec{v}_1$ and $L(\vec{v}_2) = -\vec{v}_2$.

(a) Describe this transformation geometrically.

The vectors $\vec{v}_1$ and $\vec{v}_2$ are orthogonal and hence are linearly independent. They therefore form a basis of $\mathbb{R}^2$. Since $L$ leaves $\vec{v}_1$ fixed and sends $\vec{v}_2$ to $-\vec{v}_2$, it is clear that $L$ reflects $\mathbb{R}^2$ through the straight line passing through the origin in the direction of $\vec{v}_1$. For if $\vec{x}$ is any vector in $\mathbb{R}^2$, we write $\vec{x} = a_1 \vec{v}_1 + a_2 \vec{v}_2$. Thus,

$$L(\vec{x}) = a_1 L(\vec{v}_1) + a_2 L(\vec{v}_2) = a_1 \vec{v}_1 - a_2 \vec{v}_2.$$ 

(b) Find the matrix representation of $L$ with respect to the standard basis of $\mathbb{R}^2$.

We first need to write $\vec{e}_1$ and $\vec{e}_2$ as linear combinations of $\vec{v}_1$ and $\vec{v}_2$. The matrix

$$P = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$

is the transition (change of basis) matrix from the basis $\{\vec{v}_1, \vec{v}_2\}$ to the standard basis. Thus, the columns of $P^{-1}$ will be the coordinates of the standard basis with respect to the basis $\{\vec{v}_1, \vec{v}_2\}$.

$$P^{-1} = \frac{1}{10} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$$

Thus,

$$L(\vec{e}_1) = \frac{1}{10} L(\vec{v}_1) + \frac{3}{10} L(\vec{v}_2) = \frac{1}{10} \vec{v}_1 - \frac{3}{10} \vec{v}_2 = \frac{1}{10} (-8, -6)$$

$$L(\vec{e}_2) = -\frac{3}{10} L(\vec{v}_1) + \frac{1}{10} L(\vec{v}_2) = -\frac{3}{10} \vec{v}_1 - \frac{1}{10} \vec{v}_2 = \frac{1}{10} (-6, 8)$$

Thus, the matrix representation of $L$ with respect to the standard basis of $\mathbb{R}^2$ is

$$\begin{bmatrix} -8/10 & -6/10 \\ -6/10 & 8/10 \end{bmatrix}.$$