1. (20) Define the following:

(a) \( \int_{C} f(z) \, dz \), where \( C \) is any contour.
Suppose the contour is parametrized by the function \( z(t) = x(t) + iv(t) \) for \( a \leq t \leq b \), and \( f(z) = u + iv \). Then
\[
\int_{C} f(z) \, dz = \int_{a}^{b} \left[ u(x(t), y(t)) x'(t) - v(x(t), y(t)) y'(t) \right] \, dt \\
+ i \int_{a}^{b} \left[ u(x(t), y(t)) y'(t) + v(x(t), y(t)) x'(t) \right] \, dt
\]

(b) Domain
A domain is an open connected set of complex numbers.

(c) \( z^\alpha \)
\[
z^\alpha = e^{\alpha \log(z)} = e^{\alpha (\ln|z| + i \arg(z))}
\]

(d) \( \cos z \)
\[
\cos z = \frac{e^{iz} + e^{-iz}}{2} \\
= \cos x \cosh y - i \sin x \sinh y
\]

2. (10) What are the possible values of \( (1 + i)^{-i} \)?
\[
(1 + i)^{-i} = e^{-i \log(1 + i)} = e^{-i (\ln \sqrt{2} + i(\pi/4 + 2n\pi))} \\
= e^{(\pi/4 + 2n\pi) - i \ln \sqrt{2}} \\
= e^{\pi/4 + 2n\pi} \left( \cos \left( \ln \sqrt{2} \right) - i \sin \left( \ln \sqrt{2} \right) \right)
\]
3. (10) Let $C$ denote the contour, which goes from $z = 2$ to $z = -2$ and consists of that part of the square in the upper half plane whose sides lie on the lines $x = \pm 2$, and $y = \pm 2$. Let $f(z) = \frac{1}{z^2 - i}$. Is the following inequality true or false? Explain your answer.

$$\left| \int_C f(z) \, dz \right| \leq \frac{8}{3}.$$  

The inequality is true. One way to see this is to note that for $z$ on the contour $C$ we have

$$|z^2 - i| \geq |z|^2 - 1 \geq 4 - 1 = 3.$$  

Thus, for $z$ on the contour we have $|f(z)| \leq \frac{1}{3}$. Moreover the length of the contour is 8. Hence we have

$$\left| \int_C f(z) \, dz \right| \leq \frac{1}{3} \text{length}(C) = \frac{8}{3}.$$  

4. (10) Let $C$ be the contour which consist of the following curves: the lower half of the unit circle as it goes from $z = -1$ to $z = 1$, and then the straight line segment going from $z = 1$ to $z = 3$. Calculate the value of $\int_C (2z - 1) \, dz$.

$$\int_C (2z - 1) \, dz = (z^2 - z)\big|_{-1}^3 = 6 - 2 = 4$$
5. (20) Let $C$ be the contour which consist of the following curves: the lower half of the unit circle as it goes from $z = -1$ to $z = 1$, and then the straight line segment going from $z = 1$ to $z = 3$. Calculate the value of $\int_C \log(z) \, dz$, where $\log z$ is that branch of the log function for which $\frac{\pi}{2} < \arg z \leq \frac{5\pi}{2}$. You do not need to evaluate the integrals, but be sure to evaluate the log function on the path of integration.

Let $C_1$ be that part of $C$ which consists of the lower half of the circle, and let $C_2$ be that part of $C$ which is the straight line segment from $z = 1$ to $z = 3$. These two contours are parametrized as follows

$C_1 \quad : \quad z(t) = e^{it}, \quad \pi \leq t \leq 2\pi$

$C_2 \quad : \quad z(t) = t, \quad 1 \leq t \leq 3$

Then we have

\[
\int_C \log(z) \, dz = \int_{C_1} \log(z) \, dz + \int_{C_2} \log(z) \, dz
\]

\[
= \int_{\pi}^{2\pi} \log(e^{it}) \, i e^{it} \, dt + \int_{1}^{3} \log(t) \, dt
\]

\[
= \int_{\pi}^{2\pi} (it) \, i e^{it} \, dt + \int_{1}^{3} (\ln t + 2\pi i) \, dt
\]

\[
= (-2 + 3\pi i) + (3 \ln 3 - 2 + 4\pi i)
\]

\[
= -4 + 3 \ln 3 + 7\pi i
\]
6. (30) Let \( C \) be the contour which consists of the circle centered at the origin of radius 4 traversed in a positively oriented direction. That is, in this case, a counterclockwise direction. Evaluate the following integrals:

\[
\begin{align*}
(a) & \quad \frac{1}{2\pi i} \int_C \frac{e^z - 1}{z} \, dz \\
(b) & \quad \frac{1}{2\pi i} \int_C \frac{e^z}{z(z-3)^2} \, dz
\end{align*}
\]

Solution for a.

The function \( e^z - 1 \) is an entire function. Thus, the Cauchy integral formula says

\[
\frac{1}{2\pi i} \int_C e^z - 1 \, dz = \frac{1}{2\pi i} \int_C e^z - 1 \, dz
\]

\[
= (e^z - 1)|_{z=0} = 0
\]

Solution for b.

The integrand \( \frac{e^z}{z(z-3)^2} \) is analytic at all points except \( z = 0 \) and \( z = 3 \), and both of these points lie inside the contour of integration. The Cauchy-Goursat theorem implies the integral around \( C \) is equal to the sum of two separate contour integrals. One, \( C_1 \), a small circle about \( z = 0 \), and the second, \( C_2 \), a small circle about \( z = 3 \), with both of these closed contours oriented in the positive direction. Small in this situation means that \( z = 3 \) is not inside \( C_1 \) and \( z = 0 \) is not inside \( C_2 \).

The function \( \frac{e^z}{(z-3)^2} \) is analytic inside and on \( C_1 \), while the function \( \frac{e^z}{z} \) is analytic inside and on \( C_2 \). We will use Cauchy’s integral formula on each of the two integrals.

\[
\frac{1}{2\pi i} \int_C \frac{e^z}{z(z-3)^2} \, dz = \frac{1}{2\pi i} \int_{C_1} \frac{e^z}{z(z-3)^2} \, dz + \frac{1}{2\pi i} \int_{C_2} \frac{e^z}{z(z-3)^2} \, dz
\]

\[
= \frac{1}{2\pi i} \int_{C_1} \frac{e^z/(z-3)^2}{z} \, dz + \frac{1}{2\pi i} \int_{C_2} \frac{e^z/z}{(z-3)^2} \, dz
\]

\[
= \left. \frac{e^z}{(z-3)^2} \right|_{z=0} + \frac{d}{dz} \left( \frac{e^z}{z} \right) \bigg|_{z=3}
\]

\[
= \frac{1}{9} + \frac{2}{9} e^3
\]