1. (25) Define or state the following:

(a) the completeness property for real numbers

If $E$ is a nonempty set of real numbers, which is bounded above, then there is a real
number that is a least upper bound of the set $E$.

(b) $f$ is uniformly continuous on the interval $I$

To say that $f$ is uniformly continuous on $I$ means that for every $\epsilon > 0$ there is a $\delta > 0$
such that if $x$ and $y$ belong to $I$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

(c) $f$ is Riemann integrable on the interval $[a, b]$

First $f$ is bounded on the interval $[a, b]$, and secondly for any $\epsilon > 0$ there is a partition
$P$ of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon,$$

where $U$ and $L$ denote upper and lower Riemann sums respectively.

(d) $f_n$ converges uniformly to $f$ on the set $E$

Means that for any $\epsilon > 0$ there is an $N$ such that if $n > N$, then

$$|f_n(x) - f(x)| < \epsilon$$

for every $x \in E$.

(e) the series $\sum_{n=1}^{\infty} a_n$ converges to the real number $A$.

Let $S_n = \sum_{j=1}^{n} a_j$. Then to say the infinite series converges to $A$ means that

$$\lim_{n \to \infty} S_n = A.$$
2. (25) Suppose that \( f \) is continuous on the closed bounded interval \([a, b]\).

(a) Show that \( f \) is bounded on \([a, b]\).

The proof of this will be by contradiction. So suppose that \( f \) is not bounded on the interval \([a, b]\). Then there exists a sequence of points \( x_n \in [a, b] \) such that

\[
\lim_{n \to \infty} f(x_n) = \infty.
\]

By the Bolzano-Weierstrass theorem there is a subsequence \( x_{n_i} \) of the sequence \( x_n \) that converges to some point \( x_\infty \in [a, b] \). Since \( f \) is continuous on \([a, b]\) we know

\[
\lim_{x \to x_\infty} f(x) = f(x_\infty)
\]

and \( f(x_\infty) \) is finite. However, we have \( \lim_{i \to \infty} f(x_{n_i}) = \infty \). This contradiction tells us that \( f \) must be bounded. This argument assumes that \( f \) is not bounded above. If \( f \) is bounded above, then since \( f \) is not bounded there will be a sequence \( x_n \) such that \( \lim_{n \to \infty} f(x_n) = -\infty \). In this case one argues in a similar fashion to get a contradiction.

(b) Let \( m = \inf_{x \in [a, b]} f(x) \). Show there is a point \( x_m \in [a, b] \) such that \( f(x_m) = m \).

There is a sequence of points \( x_n \in [a, b] \) such that \( \lim_{n \to \infty} f(x_n) = m \). By the Bolzano-Weierstrass theorem this sequence has a subsequence \( x_{n_i} \) that converges to a point \( x_m \in [a, b] \). By the continuity of \( f \) at the point \( x_m \) we have

\[
f(x_m) = \lim_{i \to \infty} f(x_{n_i}) = \lim_{n \to \infty} f(x_n) = m.
\]

(c) Give an example of a function \( f \) continuous on \((0, 1)\) such that \( f \) does not attain its infimum, \( m \), at any point in the set \((0, 1)\).

Set \( f(x) = x \). Then \( \inf_{x \in (0,1)} f(x) = 0 \), and for no \( x \in (0,1) \) is \( f(x) = 0 \).

3. (10) Suppose that \( f \) and \( g \) are two real valued functions defined on a set \( E \). Let \( M_f = \sup_{x \in E} f(x) \) and \( M_g = \sup_{x \in E} g(x) \). Suppose also that for every \( x \in E \) we have \( |f(x) - g(x)| \leq 1 \). Show that

\[
|M_f - M_g| \leq 1.
\]

Suppose, for example, that \( M_f - M_g > 1 \). Let \( \epsilon > 0 \) such that \( M_f - M_g > 1 + \epsilon \). Let \( x_0 \) be such that

\[
f(x_0) > M_f - \epsilon.
\]

Note that \(-g(x_0) \geq -M_g \). Then we have the following contradiction

\[
1 \geq f(x_0) - g(x_0) \geq M_f - \epsilon - M_g > 1.
\]
4. (20) Let \( x_n \) and \( y_n \) for \( n = 1, 2, \ldots \) be two sequences of real numbers. Suppose \( \lim_{n \to \infty} x_n = L = \lim_{n \to \infty} y_n \). Use the definition of the limit of a sequence of real numbers to prove the following:

(a) \( \lim_{n \to \infty} (x_n - y_n) = 0 \)

Let \( \epsilon > 0 \). Pick \( N \) such that for \( n > N \) we have both \( |x_n - L| \) and \( |y_n - L| \) less than \( \epsilon/2 \). Then for \( n > N \) we have

\[
|x_n - y_n| = |x_n - L + L + y_n| \leq |x_n - L| + |L + y_n| < \epsilon.
\]

Thus, \( \lim_{n \to \infty} (x_n - y_n) = 0 \).

(b) let \( z_n \) be the sequence obtained by interweaving the members of the sequences \( x_n \) and \( y_n \). That is,

\[
z_n = \begin{cases} x_n & \text{if } n \text{ is odd} \\ y_n & \text{if } n \text{ is even} \end{cases}
\]

Show that \( \lim_{n \to \infty} z_n = L \).

Let \( \epsilon > 0 \). Pick \( N \) such that for \( n > N \) we have both \( |x_n - L| \) and \( |y_n - L| \) less than \( \epsilon \). Then for \( n > N \) we have

\[
|z_n - L| = \begin{cases} |x_n - L| & \text{if } n \text{ is odd} \\ |y_n - L| & \text{if } n \text{ is even} \end{cases} < \epsilon.
\]

Thus, \( \lim_{n \to \infty} z_n = L \).
5. Let \( f_n(x) = \frac{x + n}{n} \) for \( x \in R \).

(a) For each fixed \( x \in R \) determine \( \lim_{n \to \infty} f_n(x) \). Denote this value by \( f(x) \).

\[
\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{x + n}{n} = 1.
\]

(b) Does the sequence \( f_n \) converge to \( f \) uniformly on \([0,1]\) ?

The sequence does converge uniformly to 1 on the interval \([0,1]\). Since

\[
|f_n(x) - 1| = \left| \frac{x + n}{n} - 1 \right| = \left| \frac{x}{n} \right| = \frac{|x|}{n} \leq \frac{1}{n},
\]

for \( x \in [0,1] \), and as \( n \) gets arbitrarily large the ratio \( \frac{1}{n} \) goes to zero.

(c) Does the sequence \( f_n \) converge to \( f \) uniformly on \( R \)?

The sequence does not converge uniformly to 1 on the whole real line. As we saw above the absolute value of the difference between \( f_n(x) \) and 1 equals \( \frac{|x|}{n} \). No matter how large \( n \) is we can pick an \( x \) such that the value of this ratio is larger than any positive number. That is, the ratio cannot be made arbitrarily small.

6. Determine whether or not each of the following series diverges, converges conditionally, or converges absolutely:

(a) \( \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \)

This series converges absolutely as is seen by comparing \( \frac{\sin x}{n^2} \) to the larger terms \( \frac{1}{n^2} \).

And since the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is known to converge the series \( \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \) converges.

(b) \( \sum_{n=1}^{\infty} \left( \sqrt{n^4 + 1} - n^2 \right) \)

Since the terms of this series are positive, convergence and absolute convergence mean the same thing for this series. One way to see that this series converges is to rationalize it.

\[
\sqrt{n^4 + 1} - n^2 = \left( \sqrt{n^4 + 1} - n^2 \right) \frac{\sqrt{n^4 + 1} + n^2}{\sqrt{n^4 + 1} + n^2}
\]

\[
= \frac{(n^4 + 1) - n^4}{\sqrt{n^4 + 1} + n^2} = \frac{1}{\sqrt{n^4 + 1} + n^2}
\]

\[
= \frac{1}{n^2 \sqrt{1 + 1/n^4} + 1} \leq \frac{1}{n^2}.
\]

Since the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges, the comparison test tells us that the original series also converges.
7. (20) A necessary condition for the infinite series $\sum_{n=1}^{\infty} a_n$ to converge to a finite number is that 
\[ \lim_{n \to \infty} a_n = 0. \]

(a) Prove that this is a necessary condition.

Let $S_n = \sum_{j=1}^{n} a_j$. Supposing that the infinite series converges to a finite number is really saying that the sequence $S_n$ converges to a finite number $L$. The terms of the sequence $a_n$ can be expressed in terms of the sequence $S_n$ as $a_n = S_n - S_{n-1}$. Thus,
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} (S_n - S_{n-1}) \\
= \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1} \\
= L - L = 0.
\]

(b) Show that it is not sufficient by giving an example of a series $\sum_{n=1}^{\infty} a_n$ that diverges for which $\lim_{n \to \infty} a_n = 0$.

We know that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and $\lim_{n \to \infty} \frac{1}{n} = 0$. 