1. (20) Define or state the following:

(a) \( \lim_{n \to \infty} a_n = l \). Note, there are 3 cases here, \( l = \pm \infty \) or \( l \) finite, define the limit for each of these cases.

1. **finite** For all \( \epsilon > 0 \), \( \exists N \) such that if \( n > N \), then \( |a_n - l| < \epsilon \).

1. **positive infinity** For all \( M \) there is an \( N \) such that if \( n > N \), then \( a_n > M \).

1. **negative infinity** For all \( M \) there is an \( N \) such that if \( n > N \), then \( a_n < M \).

(b) Cauchy sequence,

A sequence \( a_n \) is Cauchy if \( \forall \epsilon > 0, \exists N \) such that if \( n, m > N \) then

\[ |a_n - a_m| < \epsilon. \]

(c) Bolzano-Weierstrass theorem,

Every bounded sequence of real numbers contains a convergent subsequence.

(d) the function \( f \) is continuous at the point \( a \).

For every \( \epsilon > 0 \), there is a \( \delta > 0 \) such that if \( |x - a| < \delta \), then \( |f(x) - f(a)| < \epsilon \).

2. (10) Suppose that \( \lim_{n \to \infty} a_n = A \) and \( \lim_{n \to \infty} b_n = B \). Show that \( \lim_{n \to \infty} (a_n + b_n) = A + B \).

Let \( \epsilon > 0 \). Then there exist \( N_1 \) and \( N_2 \) such that if \( n > N_1 \) or \( n > N_2 \) then

\[ |a_n - A| < \frac{\epsilon}{2} \text{ or } |b_n - B| < \frac{\epsilon}{2}, \]

respectively. Then for \( n > \max(N_1, N_2) \) we have

\[ |(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \]
3. (20) Let \( E \) be a bounded set of real numbers.

(a) Define the infimum of the set \( E \).

A real number \( l \) is the infimum of a set \( E \) if \( l \) is a lower bound of \( E \), that is \( l \leq x \) for all \( x \in E \), and if \( k \) is any other lower bound of \( E \), then \( k \leq l \).

(b) Show that there is a sequence of points, \( a_n \), in \( E \) such that

\[
\lim_{n \to \infty} a_n = \inf (E).
\]

Set \( s = \inf (E) \). Since \( E \) is bounded below \( s \) is a real number and for any \( \epsilon > 0 \) there is an \( x \) in \( E \) such that \( s \leq x < s + \epsilon \). For otherwise \( s + \epsilon \) is a lower bound of \( E \) strictly greater than the greatest lower bound or infimum of \( E \). Thus, for each \( n \in \mathbb{N} \) there exists an \( a_n \in E \) such that \( s \leq a_n < s + 1/n \). By the squeeze theorem we have

\[
\lim_{n \to \infty} a_n = s.
\]

4. (10) State and prove the monotone convergence theorem for decreasing sequences of real numbers.

If \( a_n \) is a monotone decreasing sequence of real numbers that is bounded below, then the sequence converges. To prove this theorem let \( \epsilon > 0 \). Then there is an \( N \) such that \( \inf \{a_n\} \leq a_N < \inf \{a_n\} + \epsilon \). Since the sequence \( a_n \) is decreasing we have for all \( m \geq N \) that \( \inf \{a_n\} \leq a_m < \inf \{a_n\} + \epsilon \). Thus, for all \( n \geq N \) we have

\[
|a_n - \inf \{a_n\}| < \epsilon.
\]

Thus, \( \lim_{n \to \infty} a_n = \inf \{a_n\} \).
5. (15) Set \( x_1 = 1 \), and \( x_{n+1} = \sqrt{2 + x_n} \) for \( n = 1, 2, \ldots \).

(a) Show that \( x_n \) converges to some number \( l \).

To see that this sequence converges we’ll show that it is an increasing sequence bounded above by 2. The first step is to show that \( 1 \leq x_n \leq 2 \) for all \( n \). It is true for \( n = 1 \), so assume it’s true for \( n \). Then we have

\[
1 \leq \sqrt{2 + x_n} \leq \sqrt{2 + 2} = 2.
\]

Since \( x_{n+1} = \sqrt{2 + x_n} \), we have an induction proof that \( 1 \leq x_n \leq 2 \) is true for all \( n \in \mathbb{N} \).

To see that this sequence is increasing we note

\[
x_{n+1}^2 = 2 + x_n \geq x_n + x_n = 2x_n \geq x_n^2.
\]

Since \( x_{n+1} = \sqrt{x_{n+1}^2} = x_n \), we conclude that the sequence is increasing. Thus, by the monotone convergence theorem we know that the sequence converges.

(b) Determine the value of \( l \).

Taking the limit of the recursive equation we have

\[
l = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} \sqrt{2 + x_n} = \sqrt{2 + l}.
\]

This implies that \( l^2 = 2 + l \), which in turn implies that \( l = -1 \) or 2. Since \( l > 0 \), we must have \( l = 2 \).

6. (15) Use the definition of limit to show that

\[
\lim_{n \to \infty} \frac{2n + 1}{3 + n} = 2.
\]

Let \( N = \frac{5}{\epsilon} \). If \( n > N \) we have

\[
\left| \frac{2n + 1}{3 + n} - 2 \right| = \left| \frac{2n + 1 - 6 - 2n}{3 + n} \right| = \left| \frac{-5}{3 + n} \right|
\]

\[
= \frac{5}{3 + n} < \frac{5}{n} \leq \frac{5}{N} = \frac{5}{(5/\epsilon)} = \epsilon.
\]
7. (10) Let \( f(x) = \begin{cases} 
2x - 1, & x < 2 \\
3, & x = 2 \\
x^2, & 2 < x 
\end{cases} \).

(a) Explain why \( f \) is continuous at every point \( x \), except \( x = 2 \).

At every point except 2, the function \( f \) is like a polynomial in \( x \). For \( x < 2 \) it’s \( 2x - 1 \), and for \( x > 2 \) it’s \( x^2 \). Such functions are continuous everywhere so \( f \) must be continuous at all \( x \neq 2 \).

(b) What are the left and right hand limits at \( x = 2 \). In addition to determining these limits, use the definition to prove one of your answers.

\[
\lim_{{x \to 2^-}} f(x) = 3 \\
\lim_{{x \to 2^+}} f(x) = 4 .
\]

The following is a verification that \( \lim_{{x \to 2^-}} f(x) = 3 \). Let \( \epsilon > 0 \). Set \( \delta = \frac{\epsilon}{2} \). Then if \( 2 - \delta < x < 2 \) we have

\[
|f(x) - 3| = |(2x - 1) - 3| \\
= |2x - 4| \\
= 2|x - 2| < 2\frac{\epsilon}{2} = \epsilon .
\]