1. (20) Define the following:

(a) open set
A set $A \subseteq E^n$ is said to be open if $\forall \bar{x} \in A$, $\exists \delta > 0$ such that $B(\bar{x}, \delta) \subseteq A$,

where $B(\bar{x}, \delta) = \{\bar{y} : \|\bar{x} - \bar{y}\| < \delta\}$.

(b) $f : D \rightarrow E^1$ is continuous at the point $\bar{x}_0 \in D$, where $D$ is an open subset of $E^n$
$f$ is continuous at the point $\bar{x}_0$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that
if $\|\bar{x} - \bar{x}_0\| < \delta$, then $|f(\bar{x}) - f(\bar{x}_0)| < \epsilon$.

(c) Let $A \subseteq E^n$. What is the frontier of $A$?
The frontier of $A$ is the set of all frontier points of $A$; a point $\bar{x}$ is a frontier point of $A$ if every neighborhood of $\bar{x}$ contains at least one point of $A$ and at least one point of $A^c$.

(d) Let $A \subseteq E^n$. What is an accumulation point of $A$?
$\bar{x}$ is an accumulation point of $A$ if every neighborhood of $\bar{x}$ contains an infinite number of points of $A$.

2. (20) State the Bolzano-Weierstrass theorem, and prove that if $A_n$, $n = 1, 2, \cdots$ is a sequence of closed non-empty decreasing sets ($A_{n+1} \subseteq A_n$) with $\lim_{n \to \infty} \text{diam}(A_n) = 0$, then
$\cap_{n=1}^{\infty} A_n = \{\bar{x}_0\}$.

That is, the intersection of all the $A_n$ consists of exactly one point.

The Bolzano-Weierstrass theorem says that every bounded infinite subset of $E^n$ has an accumulation point.

To show that the intersection of the $A_n$ is non-empty it suffices to assume that each of the $A_n$ is infinite; for if one of them, say $A_N$ is finite, then for all $n > N$, $A_n$ must also be finite. If $A_N = \{a_1, a_2, \cdots, a_k\}$, then one of these $a_i$’s must be in each of the $A_n$ and hence in the intersection. So assuming that each $A_n$ is infinite pick an element $\bar{x}_1$ from $A_1$, then pick $\bar{x}_2$ from $A_2$ not equal to $\bar{x}_1$, continue in this fashion. Than is, pick $\bar{x}_n$ from $A_n$ such that $\bar{x}_n \notin \{\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{n-1}\}$. Set $A = \{\bar{x}_n\}_{n=1}^{\infty}$. Then $A$ is infinite and bounded. Let $\bar{x}_0$ be an accumulation point (apply the Bolzano-Weierstrass theorem) of $A$. Then $\bar{x}_0$ must be an accumulation point of each of the $A_n$, and since each of these sets is closed we have $\bar{x}_0 \in A_n$ for each $n$. Thus, $\bar{x}_0 \in \cap_{n=1}^{\infty} A_n$.

The last thing to verify is that this intersection consists of at most one point. So suppose $\bar{x}$ and $\bar{y}$ both belong to the intersection. Then for each $n$ we have
$\|\bar{x} - \bar{y}\| \leq \text{diam}(A_n)$,
but the diameters of the $A_n$ are converging to zero. Thus, we must have $\|\bar{x} - \bar{y}\| = 0$, or $\bar{x} = \bar{y}$. 
3. (10) Let $S$ be a compact subset of $E^n$.

(a) Define what compact means.

A set is said to be compact if every open cover of the set has a finite subcollection that is also a cover of the set.

(b) Show that $S$ must be closed and bounded.

To see that $S$ must be bounded, we note that $B(\bar{0}, n)$ is an open set and that $S \subseteq \cup_{n=1}^{\infty} B(\bar{0}, n)$. Compactness means that a finite subcollection of these $B(\bar{0}, n)$ must cover $S$. But then for some $N$ we must have

$$S \subseteq \cup_{n=1}^{N} B(\bar{0}, n) = B(\bar{0}, N).$$

Hence, $S$ is bounded.

To see that $S$ is closed suppose that $\bar{x}$ is any point not in $S$. Then the sets $G_n = \{\bar{y} : \|\bar{x} - \bar{y}\| > 1/n\}$ are open for each $n$ and

$$S \subseteq \cup_{n=1}^{\infty} G_n.$$

Compactness implies there is an $N$ such that

$$S \subseteq \cup_{n=1}^{N} G_n = G_N.$$

Thus, any point not in $S$ is an interior point of $S^c$. That is, $S^c$ must be open, which means $S$ is closed.

4. (10) Show that $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$. Is the same equality true if intersection is replaced with union?

To see that we have equality suppose $\bar{x} \in \text{int}(A \cap B)$, then for some $\delta > 0$, $B(\bar{x}, \delta) \subset (A \cap B)$. But, this says $B(\bar{x}, \delta) \subset A$ and $B(\bar{x}, \delta) \subset B$. That is, $\bar{x}$ is an interior point of both $A$ and $B$. Hence $\bar{x} \in \text{int}(A) \cap \text{int}(B)$. Now suppose that $\bar{x} \in \text{int}(A) \cap \text{int}(B)$. Then there is a $\delta_1 > 0$ and a $\delta_2 > 0$ such that

$$B(\bar{x}, \delta_1) \subset A \text{ and } B(\bar{x}, \delta_2) \subset B.$$

Let $\delta = \min \{\delta_1, \delta_2\}$. Then $B(\bar{x}, \delta) \subset A \cap B$, and $\bar{x}$ is an interior point of $A \cap B$. Thus, the two sets are equal.

To see that equality does not hold for unions, let $A = (0, 1)$ and $B = [1, 2]$. Then

$$\text{int} (A \cup B) = (0, 2) \text{ while } \text{int} (A) \cup \text{int} (B) = (0, 1) \cup (1, 2),$$

and the two sets are not equal. One can show that for arbitrary $A$ and $B$ that

$$\text{int} (A) \cup \text{int} (B) \subseteq \text{int} (A \cup B),$$

and, as the above example shows, this is the best we can do.
5. (20) For each of the following sets $A$ determine if it is open or closed. Then for each of these sets find its interior, frontier, closure, exterior, and its set of accumulation points.

(a) $A = \{ \vec{x} = (x^1, x^2) : x^1 < 2 - x^2 \}$

The set $A$ is open. To see this define $f : E^2 \to E^1$ by $f(x^1, x^2) = 2 - x^2 - x^1$. Then $f$ is continuous on $E^2$ and the set $A = f^{-1}((0, \infty))$. Since $(0, \infty)$ is an open set and $f$ is continuous, $A$ must also be open.

i. $\text{int}(A) = A$, since $A$ is open

ii. $\text{fr}(A) = \{ (x^1, x^2) : x^1 = 2 - x^2 \}$. The easiest way to see this is to note that if $\vec{x} = (x^1, x^2)$ is such that $x^1 = 2 - x^2$, then given any neighborhood about $\vec{x}$ of radius delta there are points in this ball in $A$ and in $A^c$. For example, consider a point of the form $\vec{x}_1 = (x^1 \pm \delta/2, x^2)$. Both $\vec{x}_+$ and $\vec{x}_-$ are in this ball with $\vec{x}_+$ in $A^c$ and $\vec{x}_-$ in $A$. Any point $\vec{x} = (x^1, x^2)$ for which $x^1 \neq 2 - x^2$ is either in $A$ or in the set $\{ \vec{x} = (x^1, x^2) : x^1 > 2 - x^2 \}$. Since $A$ is open, no point of $A$ can be a frontier point, and since the second set is also open (same reason that $A$ is open), none of its points can belong to the frontier of $A$.

iii. $\text{cl}(A) = \{ (x^1, x^2) : x^1 \leq 2 - x^2 \}.$

iv. $\text{ext}(A) = \{ (x^1, x^2) : x^1 > 2 - x^2 \}$. This follows from the fact that $\text{ext}(A) = E^2 - \text{cl}(A)$.

v. The set of accumulation points of $A$ is $\text{cl}(A)$. This follows from the fact that $A$ has no isolated points.

(b) $A = \{(0, y) : 0 \leq y \leq 1\} \cup \{(x, \sin (1/x)) : 0 < x \leq 1\}$.

This set is neither open nor closed. Points of the form $(0, y)$ for $-1 \leq y < 0$ are frontier points of $A$, but are not in $A$ so $A$ cannot be closed. To see that $A$ cannot be open, note that the point $(1, \sin 1) \in A$ but points of the form $(1 + \delta, \sin 1)$ are not in $A$ yet points of this form are in any ball about $(1, \sin 1)$.

i. $\text{int}(A) = \emptyset$

ii. $\text{fr}(A) = \{(0, y) : -1 \leq y \leq 1\} \cup \{(x, \sin (1/x)) : 0 < x \leq 1\}$

iii. $\text{cl}(A) = \text{fr}(A)$

iv. $\text{ext}(A) = \text{cl}(A)^c$

v. the set of accumulation points of $A$ is the same as the closure of $A$.

6. (10) Let $A$ be a subset of $E^n$. Define the distance of a point $x \in E^n$ to the set $A$ as

$$\text{dist}(x, A) = \inf_{y \in A} \| x - y \| .$$

Show that there is a sequence $x_n$ of points in $A$ such that

$$\lim_{n \to \infty} \| x - x_n \| = \text{dist}(x, A) .$$

Since $\text{dist}(x, A) = \inf_{y \in A} \| x - y \|$, for each integer $i$ there must be a point $x_i \in A$ such that $\| x - x_i \| < \text{dist}(x, A) + 1/i$. Thus we have

$$\text{dist}(x, A) = \inf_{y \in A} \| x - y \| \leq \| x - x_i \| < \text{dist}(A) + \frac{1}{i} .$$

This inequality implies that $\text{dist}(x, A) = \lim_{i \to \infty} \| x - x_i \|$. 

7. (10) Let \( f(x, y) = \frac{y}{x} \).

(a) Calculate \( \lim_{(x, y) \to (2, 1)} f(x, y) \)

The calculation of the limit is easy and we get

\[
\lim_{(x, y) \to (2, 1)} \frac{y}{x} = \frac{1}{2}.
\]

(b) Using the definition of limit verify your answer to part a.

Let \( \epsilon > 0 \), set \( \delta = \min \left\{ 1, \frac{2x}{3} \right\} \). Note: if \( \|(x, y) - (2, 1)\| < \delta \), then

\[
|x - 2| \leq \|(x, y) - (2, 1)\| < 1 \implies \frac{1}{|x|} < 1.
\]

Thus

\[
\left| \frac{y}{x} - \frac{1}{2} \right| = \left| \frac{2y - x}{2x} \right| = \left| \frac{2y - 2 + 2 - x}{2x} \right|
\]

\[
= \frac{1}{2|x|} |2(y - 1) + (2 - x)|
\]

\[
< |y - 1| + \frac{|2 - x|}{2}
\]

\[
\leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]