Problem 1. Let $M^n$ be a smooth manifold of dimension $m$.

a: Explain why the connected components of $M$ are the same as the path-components.

b: Explain why $M$ has at most countably many connected components.

c: Suppose that $M$ is connected, and let $p, q$ be points in $M$. Show that there is a $C^\infty$ curve $\gamma : [0, 1] \rightarrow M$ with $\gamma(0) = p$ and $\gamma(1) = q$.

Problem 2. Let $U := \{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of a smooth manifold $M$. Show that $U$ has an open refinement $V = \{V_\alpha\}_{\alpha \in \Lambda}$ which covers $M$ and satisfies $\overline{V_\alpha} \subset U_\alpha$, for all $\alpha \in \Lambda$.

HINT: Recall that if $\{W_\lambda\}$ is a locally finite family of subsets of a topological space, then

$\bigcup_\lambda \overline{W_\lambda} = \overline{\bigcup_\lambda W_\lambda}$.

Problem 3. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x, y) = x^3 + xy + y^3 + 1$. For which points $p = (0, 0)$, $p = (1/3, 1/3)$, $p = (-1/3, -1/3)$ is $f^{-1}(f(p))$ an embedded submanifold of $\mathbb{R}^2$?

Problem 4. Let $X$ be a copy of the real line $\mathbb{R}$, and let $\phi : X \rightarrow \mathbb{R}$ be $\phi(x) = x^3$. Taking $\phi$ as a chart, this defines a smooth structure on $X$. Prove or disprove:

a: $X$ is diffeomorphic to $\mathbb{R}$;

b: the identity map $X \rightarrow \mathbb{R}$ is a diffeomorphism;

c: $\phi$ together with the identity map comprise an atlas.

Problem 5. If $M \subset \mathbb{R}^n$ is a smoothly embedded manifold and $f$ is a smooth real valued function defined on a neighborhood of $p \in M$ in $\mathbb{R}^n$ which is constant on $M$. Show that the gradient $\nabla f$ is perpendicular to $T_p M$.

(REMARK: The gradient of $f$ at $x \in \mathbb{R}^n$ is the vector $\nabla f(x) = (\frac{\partial f}{\partial x_1}|_x, \ldots, \frac{\partial f}{\partial x_n}|_x)$.

Note also that $T_p M$ becomes a linear subspace of $T_p \mathbb{R}^n$ and that the latter can be identified with $\mathbb{R}^n$ itself.)
Problem 6. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a smooth curve in the plane. Let $K$ be the set of all $r \in \mathbb{R}$ such that the circle of radius $r$ about the origin is tangent to the curve $\gamma$ at some point. Show that $K$ has empty interior.

(HINT: Sard may help you.)

Problem 7. Let $f : M \to N$ be a continuous map from a space $M$ into a connected smooth $n$-dimensional manifold $N$. Suppose that every point $y \in N$ has a neighborhood $V$ such that $f^{-1}(V)$ is a disjoint union $U_1 \bigsqcup \cdots \bigsqcup U_k$ of open subsets of $M$ with the property that the restriction $f|_{U_i} : U_i \to V$ is a homeomorphism.

a: Show that $f$ is both an open map and an identification map.
b: Show that the number $k$ in the disjoint union above is the same for every point $y \in N$.
c: Show that $f$ induces a structure of smooth manifold of dimension $n$ on $M$ so that $f$ is a smooth map and a local diffeomorphism. (In particular, $f$ is a submersion and an immersion, but $f$ is not an embedding.)

Problem 8. Let $M^\mathrm{max.rk.}_{r,n} \subset M_{r,n}$ be the set of the matrices of maximal rank in $M_{r,n}$. Let $\rho : M^\mathrm{max.rk.}_{r,n} \to G_{r,n}(\mathbb{R})$ be the map which sends a matrix $M$ to the linear subspace of $\mathbb{R}^n$ generated by the rows of $M$. Show that $\rho$ is an open smooth map.

(REMARK: The manifold structure on $M^\mathrm{max.rk.}_{r,n}$ is that of an open subset of the Euclidean space $M_{r,n}$, and the manifold structure on $G_{r,n}(\mathbb{R})$ is the one described in your last assignment.

The manifold $M^\mathrm{max.rk.}_{r,n}$ is sometimes called the manifold of $r$-frames in $\mathbb{R}^n$ and denoted $F(r,n)$ or $F_{r,n}(\mathbb{R})$.)

Problem 9. Let $SL_n(\mathbb{R})$ denote the set of $n \times n$ real matrices with determinant 1. Show that $SL_n(\mathbb{R})$ is a smooth manifold and compute its dimension. Explain why $SL_n(\mathbb{R})$ is a Lie Group.

Problem 10. Let $P_{r,n}(\mathbb{R})$ be the set of $r \times n$ real matrices $A$ satisfying $A \cdot A^t = I$.

a: Identify $T_A P_{r,n}(\mathbb{R})$ with a subset of the set of all $r \times n$ matrices and verify that this subset is indeed a vector space.
b: Recall that $P_{r,n}(\mathbb{R})$ is the orthogonal group $O(n)$. Identify the tangent space $T_I O(n)$ to $O(n)$ at the identity matrix $I$.  

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