Assignment # 6

In this assignment you will learn about various constructions of classical examples of manifolds. You may feel that some details are missing, and your task is to fill those gaps. Pay special attention to the Remarks and Examples. You do not need to hand anything, but you should discuss the examples as much as possible with your classmates and your instructor.

1. Review of linear algebra

Let $V$ be an $n$-dimensional vector space over $\mathbb{R}$. The choice of a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ for $V$ determines an isomorphism

$$\phi_{\mathcal{B}} : \mathbb{R}^n \cong V$$

by sending $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ to $\phi_{\mathcal{B}}(\mathbf{x}) = x_1 v_1 + \cdots + x_n v_n$. Its inverse $\phi_{\mathcal{B}}^{-1}(v) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is denoted by $[v]_{\mathcal{B}} = (x_1, \ldots, x_n)$ and the elements of this $n$-tuple are called the coordinates of $v$ in the basis $\mathcal{B}$.

**Remark 1.** Observe that the $\phi_{\mathcal{B}}^{-1}$ can be seen as a chart on $V$, which makes it into a smooth manifold diffeomorphic to $\mathbb{R}^n$.

Consider two vector spaces $V$ and $W$ of dimensions $n$ and $m$, respectively. Denote by $\text{Hom}(V, W)$ the set of linear maps from $V$ and $W$, and recall that this is a vector space of dimension $nm$. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ and $\mathcal{B}' = \{w_1, \ldots, w_m\}$, be bases for $V$ and $W$, respectively. To a linear map $T : V \to W$, one associates an $m \times n$ matrix $[T]_{\mathcal{B}'}^B = (a_{i,j})$ defined by

$$T(v_j) = \sum_{i=1}^n a_{i,j} w_i.$$ 

This establishes an isomorphism $\Phi_{\mathcal{B}'}^\mathcal{B} : \text{Hom}(V, W) \to \mathcal{M}_{m,n}(\mathbb{R})$ between $\text{Hom}(V, W)$ and the vector space $\mathcal{M}_{m,n}(\mathbb{R})$ of $m \times n$ matrices with real entries.

Given vector spaces $L_0, \Lambda_1$ and $\Lambda_2$ of dimensions $r, m$ and $n$, respectively, the operation of composition gives a bilinear map

$$\text{Hom}(L, \Lambda_1) \times \text{Hom}(\Lambda_1, \Lambda_2) \to \text{Hom}(L, \Lambda_2)$$

$$(\phi, \psi) \mapsto \psi \circ \phi.$$ 

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1Essentially, all arguments used in this discussion apply to vector spaces over $\mathbb{C}$. You should read the discussion a second time assuming that you are dealing with vector spaces over $\mathbb{C}$, while making the appropriate changes.
If one chooses bases $B_0$, $B_1$ and $B_2$ for $L_0$, $L_1$ and $L_2$, respectively, then this operation becomes matrix multiplication $\psi \circ \phi_{B_0} = [\psi]_{B_2} \cdot [\phi]_{B_1}$. In other words, the operation above is equivalent to the smooth map:

\[
M_{m,r}(\mathbb{R}) \times M_{n,m}(\mathbb{R}) \to M_{n,r}(\mathbb{R}) \\
(A, B) \mapsto BA.
\]

In particular, given fixed $\phi_0 \in \text{Hom}(L_0, \Lambda_1)$ and $\psi_0 \in \text{Hom}(\Lambda_1, \Lambda_2)$ one defines linear maps

\[
\phi_0^* : \text{Hom}(\Lambda_1, \Lambda_2) \to \text{Hom}(L_0, \Lambda_2) \\
\psi \mapsto \psi \circ \phi_0,
\]

and

\[
\psi_0^* : \text{Hom}(L_0, \Lambda_1) \to \text{Hom}(L_0, \Lambda_2) \\
\phi \mapsto \psi_0 \circ \phi.
\]

Given $A = (a_{i,j}) \in M_{r,n}(\mathbb{R})$, its transpose $A^t = (a^t_{i,j}) \in M_{n,r}(\mathbb{R})$ is the matrix satisfying $a^t_{i,j} = a_{j,i}$. Recall that $A \in M_{n,n}(\mathbb{R})$ is said to be symmetric if $A^t = A$. The set $M_n^{\text{sym}}(\mathbb{R})$ of symmetric $n \times n$ matrices with real entries forms a vector subspace of $M_{n,n}(\mathbb{R})$.

**Remark 2.** Check that $\dim M_n^{\text{sym}}(\mathbb{R}) = \frac{n(n+1)}{2}$.

Given $M = (m_{i,j}) \in M_{r,n}(\mathbb{R})$, define its trace to be $\text{trace } M = \sum_{i=1}^{r} m_{i,i} \in \mathbb{R}$.

Now consider the following bilinear map

\[
M_{r,n}(\mathbb{R}) \times M_{r,n}(\mathbb{R}) \to M_{r,r}(\mathbb{R}) \\
A, B \mapsto AB^t.
\]

The combination of the previous two operations defines an inner product on $M_{r,n}(\mathbb{R})$ by sending $A, B$ to

\[
\langle A, B \rangle = \text{trace } (AB^t).
\]

**Remark 3.** Verify that this is indeed an inner product on the vector space $M_{r,n}(\mathbb{R})$.

Recall that the rank of a matrix $A \in M_{r,n}(\mathbb{R})$ is the dimension of the subspace of $\mathbb{R}^n$ spanned by the rows of $A$.

**Facts 1.**

1. The rank of $A$ is also the dimension of the subspace of $\mathbb{R}^n$ spanned by the columns of $A$.
2. If one considers $A$ as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^r$ then the rank-nullity theorem states that

\[
\dim \ker A + \text{rank } A = n.
\]
3. A translation of this fact to arbitrary linear maps $T : V \rightarrow W$ between finite dimensional vector spaces is:
\[
\dim \ker T + \dim \text{Im } T = \dim V.
\]

4. The rank of $A$ is the maximal size $k$ of a non-zero $k \times k$ minor determinant of $A$. In particular, if $A$ is $r \times n$ matrix with $r \leq n$, then rank $A \leq r$.

2. Examples of manifolds

2.1. Submanifolds of matrix spaces. Recall that if $M$ is a smooth manifold and $U$ is an open subset of $M$, then $U$ is an embedded submanifold of $M$. In particular, $U$ is a manifold on its own. We will simply write $M_{r,n}$ to denote $M_{r,n}(\mathbb{R})$.

Example 1. Consider the subset $M_{r,n}^{\text{max.rank}} \subset M_{r,n}(\mathbb{R})$ consisting of the $r \times n$ matrices, $r \leq n$, with maximal rank $r$. Show that $M_{r,n}^{\text{max.rank}}$ is an open subset of $M_{r,n}$. Conclude that $M_{r,n}^{\text{max.rank}}$ is a manifold of dimension $rn$.

In the case where $r = n$, a matrix $A$ is in $M_{n,n}^{\text{max.rank}}$ if and only if $A$ is invertible.

Example 2. The set of invertible $n \times n$ matrices is an open subspace of $M_{n,n}$ and is denoted by $GL_n(\mathbb{R})$ and it is called the general linear group. Verify that that operations $GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \rightarrow GL_n(\mathbb{R})$
\[
(A, B) \mapsto AB
\]
and
\[
GL_n(\mathbb{R}) \rightarrow GL_n(\mathbb{R})
A \mapsto A^{-1}
\]
are smooth maps, and give $GL_n(\mathbb{R})$ the structure of a group. A group $G$ which is a topological space and whose operations of multiplication and inverse are continuous is called a topological group. A group $G$ which is a smooth manifold and whose operations of multiplication and inverse are smooth is called a Lie group.

From now one, assume that $r \leq n$, unless otherwise said. Consider the map
\[
(5) \quad \Phi : M_{r,n} \rightarrow M_{r}^{\text{sym}}(\mathbb{R})
A \mapsto AA^t
\]
Verify that this is a smooth map between smooth manifolds. Let us understand its differential
\[
\Phi_* : T_PM_{r,n} \rightarrow T_{\Phi(P)}M_{r}^{\text{sym}}(\mathbb{R})
\]
at a point $P \in M_{r,n}$. Remember that the manifolds in question are isomorphic (as vector spaces) to $\mathbb{R}^m$ and $\mathbb{R}^r$, hence one can interpret tangent spaces in the "old-fashioned" way. Pick $v \in T_PM_{r,n}$, and note that the previous observation allows one to consider $v$ as an $r \times n$ matrix. Define $\gamma(t) = A + tv$ and note that $\gamma(0) = A$ and $D_v = v = \gamma'(0)$. By definition, $\Phi_*(v) = \Phi_*(D_\gamma) = D_{\Phi_\gamma} = (\Phi \circ \gamma)'(0)$. However, one can verify that $\Phi \circ \gamma(t) = AA^t + t(vA^t + Av^t) + t^2vv^t$, hence

$\Phi_*(v) = (\Phi \circ \gamma)'(0) = vA^t + Av^t$.

Example 3. Define $\mathcal{P}_{r,n}(\mathbb{R}) = \{A \in M_{r,n}(\mathbb{R}) \mid AA^t = I\} = \Phi^{-1}(I)$. Our goal is to show that this is an embedded compact submanifold of $M_{r,n}^{\text{max.rk.}}$. We will do this in a few steps.

1. Show that $\mathcal{P}_{r,n}(\mathbb{R}) \subset M_{r,n}^{\text{max.rk.}}$.

Observe that if $A \in \mathcal{P}_{r,n}(\mathbb{R})$ then one sees that the composition

$$\mathbb{R}^r \xrightarrow{A^t} \mathbb{R}^n \xrightarrow{A} \mathbb{R}^r$$

is precisely the identity. This shows that as a linear map $A$ is surjective. Apply the rank-nullity theorem to conclude that rank $A = r$.

2. Show that $I$ is a regular value of $\Phi$.

This means that if $A \in \Phi^{-1}(I) = \mathcal{P}_{r,n}(\mathbb{R})$ then $\Phi_* : T_AM_{r,n} \to T_{\gamma}M_{r,n}^{\text{sym.}}(\mathbb{R})$ is surjective. This ammounts to show that given $w \in T_{\gamma}M_{r,n}^{\text{sym.}}(\mathbb{R}) = M_{r,n}^{\text{sym.}}(\mathbb{R})$ (recall this identification) one can find $v \in M_{r,n}$ such that $\Phi_*(v) = w$. However, we have seen that $\Phi_*(v) = vA^t + Av^t$. Hence if one defines $v := \frac{1}{2}wA$, then

$$\Phi_*(v) = (\frac{1}{2}wA)A^t + A(\frac{1}{2}wA)^t$$

$$= \frac{1}{2}wAA^t + \frac{1}{2}AA^tw^t = \frac{1}{2}w + \frac{1}{2}w^t$$

where the second to last inequality follows from the fact that $A$ satisfies $AA^t = I$ and the last one follows from the fact that $w$ is a symmetric matrix.

We will prove the following result in class:

**Theorem 4.** Let $\Phi : M \to N$ denote a smooth map between smooth manifolds of dimensions $m$ and $n$ respectively, and let $y \in N$ be a regular value such that $\Phi^{-1}(y) \neq \emptyset$. Then $\Phi^{-1}(y)$ is an embedded $(m - n)$-dimensional submanifold of $M$. 

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It follows from this theorem and the above observations that $P_{r,n}(\mathbb{R})$ is an embedded submanifold of $M_{r,n}^{\maxrk}$, and that its dimension is

$$\dim P_{r,n}(\mathbb{R}) = rn - \frac{r(r+1)}{2} = \frac{r}{2}(2n - r - 1).$$

3. $P_{r,n}(\mathbb{R})$ is compact.

Just observe that since $P_{r,n}(\mathbb{R}) = \Phi^{-1}(I) \subset M_{r,n}$, then $P_{r,n}(\mathbb{R})$ is a closed subspace of $M_{r,n}$. On the other hand, if $A \in P_{r,n}(\mathbb{R})$, then $|A|^2 = \langle A, A \rangle = \text{trace}(AA^t) = \text{trace} I = r$. Hence, $P_{r,n}(\mathbb{R})$ is a bounded subspace of $M_{r,n}$. This shows that $P_{r,n}(\mathbb{R})$ is compact.

Example 4. Note that when $r = n$ then every matrix in $P_{n,n}(\mathbb{R})$ is invertible, and that is is closed under matrix multiplication. We denote $P_{n,n}(\mathbb{R})$ by $O(n)$ and call it the orthogonal group. This is a compact subgroup of the general linear group $GL_n(\mathbb{R})$, and it is another example of a Lie group, since it is a manifold and a group with smooth operations. It follows from the formula above that $\dim O(n) = \frac{n(n-1)}{2}$.  

2.2. Grassmann manifolds. Let $G_{r,n}(\mathbb{R})$ denote the set of $r$-dimensional linear subspaces of $\mathbb{R}^n$. The techniques used to show the following result form a beautiful application of linear algebra combined with the constructions made above. Pay lots of attention to the details.

Proposition 5. The set $G_{r,n}(\mathbb{R})$ has a natural topology which makes it into a compact smooth manifold of dimension $r(n - r)$.

Remark 6. Note that $G_{1,n+1}(\mathbb{R})$ is the set of lines through the origin in $\mathbb{R}^{n+1}$. There is a surjective map $S^n \to G_{1,n+1}(\mathbb{R})$ which sends $x \in S^n$ to the line containing $x$ and the origin. You have seen this before. In fact, $G_{1,n+1}(\mathbb{R})$ is nothing else than the projective space $\mathbb{R}P^n$.

1. Topologizing $G_{r,n}(\mathbb{R})$.

This follows the idea in the previous remark. Consider the manifold $P_{r,n}(\mathbb{R})$ studied above. Given $A \in P_{r,n}(\mathbb{R})$ we know that the subspace $\text{span}(A)$ of $\mathbb{R}^n$, generated by the rows of $A$, has dimension $r$. Hence this defines a surjective map

$$\pi : P_{r,n}(\mathbb{R}) \to G_{r,n}(\mathbb{R}).$$

Give $G_{r,n}(\mathbb{R})$ the quotient topology and observe that $G_{r,n}(\mathbb{R})$ becomes a compact space.

2. Understanding the map $\pi$.

Given a linear subspace $L \subset \mathbb{R}^n$ of dimension $r$, the fiber $\pi^{-1}(L)$ of the map $\pi$ consist of the $r \times n$ matrices $A \in P_{r,n}(\mathbb{R})$ whose rows span $L$.

Prove the following.
Lemma 7. If $A, B \in \pi^{-1}(L)$ then there is a unique orthogonal matrix $M \in O(r)$ so that $A = MB$.

Note that matrix multiplication gives a map $\xi : O(r) \times \mathcal{P}_{r,n}(\mathbb{R}) \rightarrow \mathcal{P}_{r,n}(\mathbb{R})$, defined by $\xi(M, A) = MA$. Now, given a subset $W \subset \mathcal{P}_{r,n}(\mathbb{R})$, denote $O(r) \cdot W$ the image of $O(r) \times W$ under $\xi$.

Proof:
1. If $W \subset \mathcal{P}_{r,n}(\mathbb{R})$ is open, then $O(r) \cdot W$ is open.
2. If $W \subset \mathcal{P}_{r,n}(\mathbb{R})$ is closed, then $O(r) \cdot W$ is closed.

3. $G_{r,n}(\mathbb{R})$ is Hausdorff.

Given $L \in G_{r,n}(\mathbb{R})$ the lemma above shows that $\pi^{-1}(L)$ is homeomorphic to the compact Lie group $O(r)$. Hence, $\pi^{-1}(L)$ is a closed subspace of $\mathcal{P}_{r,n}(\mathbb{R})$. This shows that $G_{r,n}(\mathbb{R})$ is $T_1$.

Let $U$ be an open neighborhood of $L \in G_{r,n}(\mathbb{R})$. We will show that it contains an open neighborhood $V$ of $L$ such that $L \in V \subset \overline{V} \subset U$. Since $G_{r,n}(\mathbb{R})$ is $T_1$, this suffices to show that $G_{r,n}(\mathbb{R})$ is Hausdorff.

The inverse image $\pi^{-1}(U)$ is an open subset of $\mathcal{P}_{r,n}(\mathbb{R})$ containing the closed subset $\pi^{-1}(L)$. Since $\mathcal{P}_{r,n}(\mathbb{R})$ is normal (why?) one finds an open set $W$ such that $\pi^{-1}(L) \subset W \subset \overline{W} \subset \pi^{-1}(U)$.

Use the previous item to verify that $O(r) \cdot W \subset O(r) \cdot \overline{W} \subset \pi^{-1}(U)$.

Furthermore, show that $V = \pi(O(r) \cdot W)$ is an open set satisfying $V \subset \overline{V} \subset U$.

3. Locally Euclidean.

Let $\Lambda \subset \mathbb{R}^n$ be an $(n-r)$-dimensional subspace. Define

$$U_\Lambda := \{L \in G_{r,n}(\mathbb{R}) \mid L \cap \Lambda = \{0\}\}.$$ 

Verify that $L \in U_\Lambda$ if and only if the map $j : L \oplus \Lambda \rightarrow \mathbb{R}^n$ defined by $j(v, w) = v + w$ is an isomorphism.

Lemma 8. Fix an element $L_0 \in U_\Lambda$, and let $i_0 : \mathbb{R}^n \rightarrow L_0 \oplus \Lambda$ denote the inverse of the isomorphism above. Then $U_\Lambda$ is homeomorphic to $\text{Hom}(L_0, \Lambda)$.

In particular, $U_\Lambda$ is homeomorphic to $\mathbb{R}^{(n-r)}$.

Proof. Given $L \in U_\Lambda$, then $i_0(L)$ is a linear subspace of $L_0 \oplus \Lambda$ which is seen to be the graph of a unique linear map $\phi_L : L_0 \rightarrow \Lambda$. This will follow from the definition of $U_\Lambda$.

More precisely, consider the projections $\pi_1 : L_0 \oplus \Lambda \rightarrow L_0$ and $\pi_2 : L_0 \oplus \Lambda \rightarrow \Lambda$. Given $L \subset \mathbb{R}^n$, consider the restriction $(\pi_1 \circ i_0)|_L : L \rightarrow L_0$.

Observe that since $L \in U_\Lambda$ then this restriction is an isomorphism. Now, consider the map $\phi_L := (\pi_2 \circ i_0) \circ (\pi_1 \circ i_0)^{-1} : L_0 \rightarrow \Lambda$. Then

$$(x, y) \in \text{graph}(\phi_L)$$

if and only if

$$y = (\pi_2 \circ i_0) \circ (\pi_1 \circ i_0)^{-1}(x).$$
if and only if
\[ x = (\pi_1 \circ i_0)|_L(u) \] and \( y = \pi_2 \circ i_0(u) \) for a unique \( u \in L \),
if and only if
\[ (x, y) = j(u) \]
for a unique \( u \in L \). It is easy to see now that the map \( L \in U \Lambda \mapsto \phi_L \) defines a homeomorphism between \( U \Lambda \) and \( Hom(L_0, \Lambda) \).

Remark 9. At this point, it is not very difficult for you to prove that \( G_{r,n}(\mathbb{R}) \) is 2nd countable.

Given \( L_0 \in U_{\Lambda_1} \cap U_{\Lambda_2} \), consider the isomorphisms \( i_1 : \mathbb{R}^n \to L_0 \oplus \Lambda_1 \) and \( i_2 : \mathbb{R}^n \to L_0 \oplus \Lambda_2 \) as in the proof of the lemma. Verify that the composition
\[ L_0 \oplus \Lambda_1 \xrightarrow{i_1^{-1}} \mathbb{R}^n \xrightarrow{i_2} L_0 \oplus \Lambda_2 \]
has the form \( i_2 \circ i_1^{-1}(v, w) = (v, \Psi_{12}(w)) \), where \( \Psi_{12} : \Lambda_1 \to \Lambda_2 \) is an isomorphism.

Now, let us denote by \( \phi_i : U_{\Lambda_i} \to Hom(L_0, \Lambda_i) \), for \( i = 1, 2 \), the isomorphisms described in the Lemma above. It follows from the above considerations that the composition
\[ \phi_2 \circ \phi_1^{-1} : \phi_1(U_{\Lambda_1} \cap U_{\Lambda_2}) \to \phi_2(U_{\Lambda_1} \cap U_{\Lambda_2}) \]
is given by taking \( \varphi \in \phi_1(U_{\Lambda_1} \cap U_{\Lambda_2}) \subset Hom(L_0, \Lambda_1) \) to \( \Psi_{12} \circ \varphi \in Hom(L_0, \Lambda_2) \). (Verify this assertion.) In other words, \( \phi_2 \circ \phi_1^{-1} \) is the restriction of \( \Psi_{12} : Hom(L_0, \Lambda_1) \to Hom(L_0, \Lambda_2) \) to the appropriate subsets. Since this is a linear map, it is smooth.

In conclusion, the collection \( \{(U_{\Lambda}, \phi_{\Lambda})\} \) forms an atlas for \( G_{r,n}(\mathbb{R}) \). Therefore, \( G_{r,n}(\mathbb{R}) \) is a compact, smooth manifold of dimension \( r(n - r) \).