Problem List # 1

Classical Algebraic Geometry

Remark: Assume the ground field $k$ is algebraically closed unless stated otherwise.

Problem 1. a. Let $Y$ be the plane curve $y = x^2$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over $k$.

b. Let $Z$ be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over $k$.

Problem 2. (The twisted cubic curve) Let $Y \subset \mathbb{A}^3$ be the set

$$Y = \{(t, t^2, t^3) \mid t \in k\}.$$ 

Show that $Y$ is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over $k$. (We say that $Y$ is given by the “parametric representation” $x = t, y = t^2, z = t^3$.)

Problem 3. Let $Y$ be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that $Y$ is a union of three irreducible components. Describe them and find their prime ideals.

Problem 4. If we identify $\mathbb{A}^1 \times \mathbb{A}^1$ with $\mathbb{A}^2$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

Problem 5. Let $\mathfrak{A} \subset k[x_1, \ldots, x_m]$ be an ideal which can be generated by $r$ elements. Then every irreducible component of $Z(\mathfrak{A})$ has dimension $\geq n - r$.

Problem 6. Let $Y \subset \mathbb{A}^3$ be the curve given parametrically by $x = t^3, y = t^4, z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $k[x, y, z]$ which cannot be generated by 2 elements. (We say $Y$ is not a local complete intersection.)

Sheaves

Problem 7. Given a morphism $\phi : \mathcal{F} \to \mathcal{G}$ of sheaves of abelian groups on a space $X$, we define a presheaf $\ker \phi$ on $X$ by $(\ker \phi)(U) = \ker \{\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)\}$.
a. Show that ker \( \phi \) is indeed a sheaf.

b. Show that \((\ker \phi)_P = \ker \phi_p\), and that \( \phi \) is injective (surjective) if and only if the induce maps on stalks is injective (surjective).

c. Formulate similar notions of a presheaf coker \( \phi \) and image \( \phi \), and show that they are not necessarily a sheaf, in general.

**Problem 8.** *(Espace étalé of a presheaf)* Given a presheaf \( \mathcal{F} \) on \( X \), we define a topological space \( \text{Spé}(\mathcal{F}) \), called the *espace étalé* of \( \mathcal{F} \), as follows. As a set, \( \text{Spé}(\mathcal{F}) = \bigcup_{P \in X} \mathcal{F}_P \). We define a projection map \( \pi : \text{Spé}(\mathcal{F}) \to X \) by sending \( s \in \mathcal{F}_P \) to \( P \in X \). For each open set \( U \subseteq X \) and each section \( s \in \mathcal{F}(U) \), we obtain a map \( \overline{s} : U \to \text{Spé}(\mathcal{F}) \) for all \( U \) by sending \( P \mapsto s_P \), its germ at \( P \). This map has that property that \( \pi \circ \overline{s} = \text{id}_U \), i.e. it is a section of \( \pi \) over \( U \). We now make \( \text{Spé}(\mathcal{F}) \) into a topological space by giving it the finest topology making all maps \( \overline{s} : U \to \text{Spé}(\mathcal{F}) \), for all \( s \in \mathcal{F}(U) \) and all \( U \subseteq X \), continuous.

a. Define \( \mathcal{F}^+ \) as the presheaf where \( \mathcal{F}^+(U) \) is the set of all continuous sections of \( \pi \) over \( U \). Show that \( \mathcal{F}^+ \) is a sheaf.

b. Show that if \( \mathcal{F} \) is itself a sheaf then \( \mathcal{F}^+ = \mathcal{F} \).

**Remark:** The sheaf \( \mathcal{F}^+ \) is called the sheaf associated to the presheaf \( \mathcal{F} \).

**Problem 9.** *(Skyscraper Sheaves)* Let \( P \) be a point in a space \( X \), and let \( A \) be an abelian group. Define a sheaf \( i_P(A) \) on \( X \) as follows: \( i_P(A)(U) = A \) if \( P \in U \) and \( i_P(A)(U) = \{0\} \), otherwise.

a. Verify that the stalk of \( i_P(A) \) is \( A \) at every point \( Q \in \overline{\{P\}} \) and \( \{0\} \) elsewhere.

b. Show that this sheaf can also be described as \( i_\ast(A) \), where \( A \) is the constant sheaf on the closed subspace \( \{P\} \), and \( i : \{P\} \to X \) is the inclusion map.

**Problem 10.** Exercise I-10, p. 15.

**Schemes**

**Problem 11.** Describe \( \text{Spec}(\mathbb{Z}) \) and show that each scheme \( X \) admits a unique morphism to \( \text{Spec}(\mathbb{Z}) \). In other words, \( \text{Spec}(\mathbb{Z}) \) is a final object in the category of schemes.
Problem 12. Let $X$ be a scheme, and recall that the residue field of a point $P \in X$ is the field $k(P) = \mathcal{O}_{X,P}/\mathfrak{m}_P$. Now, let $K$ be any field. Show that to give a morphism of $\text{Spec}(K)$ to $X$ it is equivalent to give a point $P \in X$ and an inclusion homomorphism $k(P) \to K$.

Problem 13. If $X$ is a topological space and $Z$ is an irreducible closed subset of $X$, a generic point for $Z$ is a point $\xi$ such that $Z = \overline{\{\xi\}}$. If $X$ is a scheme, show that every (nonempty) irreducible closed subset has a unique generic point.

Problem 14. Let $A$ be a ring. Show that the following conditions are equivalent:

a. $\text{Spec}(A)$ is disconnected;

b. there exist nonzero elements $e_1, e_2$ of $A$ such that $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1 + e_2 = 1$. (These elements are called orthogonal idempotents).

c. $A$ is isomorphic to a direct product $A_1 \times A_2$ of two nonzero rings.
