Math 131 Week-in-Review #5 (Sections 2.7-2.9, 3.1)

1. A pie has just been taken out of the oven and is cooling before being eaten. Suppose that \( T(m) \) is the temperature of the pie, in degrees Fahrenheit, after it has been out of the oven \( m \) minutes.

   a) Interpret \( T(5) = 315 \).

   "5 minutes after being taken out of the oven, the temperature of the pie is 315°F."

   b) What are the units of \( T'(m) \), and what is the sign of \( T'(m) \)?

     \[
     \text{Output per input} \quad \Rightarrow 10^\circ F/\text{minute} \quad \Rightarrow T'(m) < 0 \text{ (negative)} \quad \text{(temp. is decreasing)}
     \]

   c) Interpret \( T'(5) = -18 \).

     "5 minutes after being taken out of the oven, the temperature is decreasing at a rate of 18°F/minute."

2. Life expectancy improved dramatically in the 20th century. The table below gives values of \( E(t) \), the life expectancy at birth (in years) of a male born in the year \( t \) in the United States. Estimate and interpret the values of \( E'(1910) \) and \( E'(1950) \). (Source: #32, pg. 154, Stewart)

<table>
<thead>
<tr>
<th>Year</th>
<th>Life Expectancy (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1900</td>
<td>48.3</td>
</tr>
<tr>
<td>1910</td>
<td>51.1</td>
</tr>
<tr>
<td>1920</td>
<td>55.2</td>
</tr>
<tr>
<td>1930</td>
<td>57.4</td>
</tr>
<tr>
<td>1940</td>
<td>62.5</td>
</tr>
<tr>
<td>1950</td>
<td>65.6</td>
</tr>
<tr>
<td>1960</td>
<td>66.6</td>
</tr>
<tr>
<td>1970</td>
<td>67.1</td>
</tr>
<tr>
<td>1980</td>
<td>70.0</td>
</tr>
<tr>
<td>1990</td>
<td>71.8</td>
</tr>
<tr>
<td>2000</td>
<td>74.1</td>
</tr>
</tbody>
</table>

*Use average rates of change to estimate!*

\[ E'(1910) \]

\[
\frac{51.1 - 48.3}{1910 - 1900} = \frac{2.8}{10} = 0.28 \text{ yrs/yr}
\]

\[ E'(1950) \]

\[
\frac{65.6 - 62.5}{1950 - 1940} = \frac{3.1}{10} = 0.31 \text{ yrs/yr}
\]

In 1910, life expectancy was increasing at a rate of 0.28 yrs/yr.

In 1950, life expectancy was increasing at a rate of 0.31 yrs/yr.
3. Use the definition of the derivative to find the derivative of \( f(x) = \sqrt{5x-1} \), and then use your formula to find the equation of the line tangent to \( f(x) \) at the point \((2, 3)\).

\[
F'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{5(x+h)-1} - \sqrt{5x-1}}{h}
\]

\[
= \lim_{h \to 0} \frac{\sqrt{5x+5h-1} - \sqrt{5x-1}}{h} \cdot \frac{\sqrt{5x+5h-1} + \sqrt{5x-1}}{\sqrt{5x+5h-1} + \sqrt{5x-1}}
\]

\[
= \lim_{h \to 0} \frac{5x+5h-1 - (5x-1)}{h(\sqrt{5x+5h-1} + \sqrt{5x-1})} = \lim_{h \to 0} \frac{5h}{h(\sqrt{5x+5h-1} + \sqrt{5x-1})} = \frac{5}{2\sqrt{5x-1}} = F'(x)
\]

Tangent line:

\[
F'(2) = \frac{5}{2\sqrt{10-1}} = \frac{5}{6} = m
\]

\[
\Rightarrow \ y - 3 = \frac{5}{6} (x - 2) \quad \text{or} \quad \ y = \frac{5}{6} x + \frac{8}{3}
\]

4. Complete the following in order to describe the relationships among \( f(x), f'(x), \) and \( f''(x) \).

\( f'(x) > 0 \) means that \( f(x) \) is increasing.

\( f'(x) < 0 \) means that \( f(x) \) is decreasing.

\( f''(x) > 0 \) means that \( f'(x) \) is increasing and \( f(x) \) is concave up.

\( f''(x) < 0 \) means that \( f'(x) \) is decreasing and \( f(x) \) is concave down.
5. Consider the graph of \( f(x) \) below.

a) Arrange the derivatives at the given points from smallest to largest.

\[ F, C, E, B, A, D \]

b) At what points do \( f'(x) \) and \( f''(x) \) have the same sign?

\[ f'(x) > 0 \text{ and } f''(x) > 0 \Rightarrow f(x) \text{ is concave up} \Rightarrow \]

\[ A, B \] (positive)

\[ f'(x) < 0 \text{ and } f''(x) < 0 \Rightarrow f(x) \text{ is concave down} \Rightarrow \]

\[ F \] (negative)

6. The equation of motion of a particle is \( s = 3t^3 - 6t^2 + 3t + 1 \), where \( s \) is in meters and \( t \) is in seconds.

Find the velocity and acceleration functions of the particle, as well as the acceleration after 1 second.

\[ v(t) = s'(t) = 9t^2 - 12t + 3 \text{ m/s} \] (velocity)

\[ a(t) = v'(t) = s''(t) = 18t - 12 \text{ m/s/s} \] (acceleration)

\[ a(1) = 6 \text{ m/s}^2 \]

Note:

\[ v(1) = 0 \text{ m/s} \Rightarrow \text{particle is at rest!} \]
7. Sketch the graph of the derivative of each of the following functions.

a)

Look for:
- concavity
- inflection points
- local max/min

\( h(x) \)

\[ h'(x) = \begin{cases} \downarrow & a \\ \uparrow & b \\ \downarrow & \text{(linear)} \end{cases} \]

\( h' \) is zero \( \Theta \) \( b \) \( \Theta \) (constant slope)

(less \( \Theta \)) (more \( \Theta \)) (hole)

*deriv. DNE @ Sharp corner!

b)

\[ k(x) \]

\[ k'(x) \]

\[ k' \] is zero \( \Theta \) \( \Theta \) (DNE) \( \Theta \)

near zero \( \rightarrow \) (more \( \Theta \)) \( \rightarrow \) (less pos) \( \rightarrow \) near zero

Vertical asymptote \( x = 0 \).
8. Find the derivative of each of the following functions.

a) \( j(x) = \frac{2e^x}{x} - \frac{\sqrt{x^3}}{x^2} + xx^3 - 1.8^3 \)  
   Rewrite: \( j(x) = -\frac{2}{5} e^x - x^3 + \pi x - 1.8^3 \)

\[ j'(x) = -\frac{2}{5} e^x - \frac{3x^2}{\sqrt{x^3}} + \pi \]

b) \( m(x) = x^8 + 3x^2 - \frac{\sqrt{x^3}}{8} + 3x \)  
   \( m(x) = \frac{x^8 + 3x^2}{8} \frac{\sqrt{x^3}}{3} + 3x \)

\[ m'(x) = -\frac{24}{24} x^7 - \frac{9x^2}{24} \frac{1}{3} - \frac{7}{96} x \frac{\sqrt{3}}{x} + (\ln 3) 3x \]

9. Sketch the graph of a function that satisfies all of the following conditions:

- \( f'(1) = f'(-1) = 0 \)  
  horizontal tangent at \( x = 1 \)  
  \( x = -1 \)
- \( f'(x) < 0 \) if \( |x| < 1 \)  
  decreasing on \((-1, 1)\)
- \( f'(x) > 0 \) if \( 1 < |x| < 2 \)  
  increasing on \((-2, -1)\) and \((1, 2)\)
- \( f''(x) = -1 \) if \( |x| > 2 \)  
  has constant slope \(-1\) on \((-\infty, -2)\) and \((2, \infty)\)
- \( f''(x) < 0 \) if \(-2 < x < 0\)  
  concave down on \((-2, 0)\)
- inflection point at \((0,1)\)

Shape of graph:

- concave up
- concave down

Function behavior:

- increasing
- decreasing

(f(x))

(slope \(-1\))
10. Let $F$ be an antiderivative of the function $f$ whose graph is shown below.

\[
\begin{array}{c}
\text{derivative of } F.
\end{array}
\]

\[
\begin{array}{c}
f(x)
\end{array}
\]

a) Where is $F$ increasing?
(Where its derivative, $f(x)$, is positive) \Rightarrow (0, 4) \text{ and } (8, \infty)

b) Where is $F$ decreasing?
(Where $f(x)$ is negative) \Rightarrow (-\infty, 0) \text{ and } (4, 8)

c) Where is $F$ concave up?
(Where its derivative, $f(x)$, is increasing) \Rightarrow (-\infty, 2) \text{ and } (6, 9)

d) Where is $F$ concave down?
(Where $f(x)$ is decreasing) \Rightarrow (2, 6) \text{ and } (9, \infty)

e) Use the above information to sketch a graph if $F(0) = 0$. 

\[
\begin{array}{c}
\text{Shape of } F.
\end{array}
\]

\[
\begin{array}{c}
\text{Incr.} \quad \text{Decr.}
\end{array}
\]

\[
\begin{array}{c}
\text{CC up/down}
\end{array}
\]

\[
\begin{array}{c}
\text{F}
\end{array}
\]

\[
\begin{array}{c}
\text{F}
\end{array}
\]