Math 131 Week-in-Review #7 (Exam 2 Review: Sections 2.7-2.9, 3.1-3.5, and 3.7-3.8)

Note: This collection of questions is intended to be a brief overview of the exam material. When studying, you should also look at your notes, the suggested homework problems from the textbook, as well as the other week-in-reviews for this material.

1. The number of reptile species found in a wetland area can be modeled by

\[ R(m) = 17 + 8 \cos \left( \frac{\pi}{6} m \right) \text{ species} \]

where \( m \) is the end of the \( m^{th} \) month of the year. Find and interpret \( R(10) \) and \( R'(10) \).

\[ R(10) = 17 + 8 \cos \left( \frac{\pi}{6} (10) \right) = 21 \text{ species} \]

\( R(10) \) at the end of October, there are 21 reptile species.

\[ R'(m) = -8 \sin \left( \frac{\pi}{6} m \right) \text{ species/month} \Rightarrow R'(10) = -8 \sin \left( \frac{\pi}{6} (10) \right) = \frac{3}{3.6276} \text{ species/month} \]

\( R'(10) \) at the end of October, the # of reptile species is increasing by 3.6276 species/month.

2. Sketch the graph of the derivative of \( f(x) \) shown below.

Solution:

\[ f'(x) \]

\[ f \]

\[ f' \]

V.A., \( x = 0 \)

Near zero (more neg) (less pos)

(Zero) DNE
3. The table below gives the average yearly health care costs (per consumer unit) for various years since 1990.

<table>
<thead>
<tr>
<th></th>
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<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Per capita expenditure ($)</td>
<td>1480</td>
<td>1732</td>
<td>1910</td>
<td>2066</td>
<td>2350</td>
</tr>
</tbody>
</table>

a) Find the average rate of change in health care costs (per consumer unit) between 1998 and 2002 and interpret your answer.

\[
\frac{2350 - 1910}{2002 - 1998} = \frac{110}{1} \text{ $ per consumer/yr.}
\]

Between 1998 and 2002, health care costs were increasing on average by $110 per consumer/yr.

b) Estimate the rate of change in health care costs (per consumer unit) in 2002 and interpret your answer.

\[
\frac{2000 + 2003 - 2350 - 2002}{2002 - 2000} = \frac{142}{2} \text{ $ per consumer/yr.}
\]

In 2002, health care costs were increasing by $142 per consumer/yr.

c) Use a linear approximation to estimate health care costs (per consumer unit) in 2005. Is your answer an overestimate or an underestimate? Why?

\[f(x) = f(a) + f'(a)(x-a), \quad x = 2005 \Rightarrow a = 2002 \quad (x \text{ close to } a!)
\]

\[f(2002) = 2350 \text{ $ per consumer}
\]

\[f'(2002) = 142 \text{ $ per consumer/yr.} \quad \Rightarrow \text{See part b above for } f'(2002) \text{ estimate.}
\]

\]

\[= 2350 + 142(2005 - 2002)
\]

\[= 2350 + 142(3) = 2776 \text{ $ per consumer}
\]

Plot data & see tangent is below curve ⇒ underestimate.
4. Use the graph below to answer questions a) - d).

(a graph courtesy of Joe Kahliq)

a) If the graph is $f(x)$, where is $f'(x)$ negative? (where $f(x)$ is decreasing)

\[ \Rightarrow (-2, 3) \]

b) If the graph is $f'(x)$, where is $f(x)$ increasing? (where $f'(x)$ is positive)

\[ \Rightarrow (-1, 0) \text{ and } (6, \infty) \]

c) If the graph is $f''(x)$, where is $f(x)$ concave up? ($f(x)$ cc up $\Rightarrow F''(x) > 0 \Rightarrow f'(x)$ increasing)

\[ \Rightarrow \text{ need where } F'(x) \text{ is increasing } \]

\[ (-\infty, -2) \text{ and } (3, \infty) \]

d) If the graph is $f''(x)$, where is $f(x)$ concave up? (where $f''(x)$ is positive)

\[ (-4, 0) \text{ and } (6, \infty) \]
5. Find the derivative of each of the following functions. Do not simplify your answers.

a) \( f(x) = \ln(4^{x^2-3x}) + 3e^{3x^2}(\tan^3(\pi x + 2)) \)

\[
\begin{align*}
  f'(x) &= \frac{1}{4x^2-3x} \left[ (\ln 4) 4^{x^2-3x}(2x-\pi) \right] + \\
  &+ 3e^{3x^2} \left[ 3 (\tan(\pi x + 2))^2 \left[ \sec^2(\pi x + 2) \right] (\pi) \right] + \\
  &+ (\tan^3(\pi x + 2)) \left[ 3 e^{3x^2} \left[ \frac{y^5}{3x^4} \left( \frac{y^5}{3x^4} \right) \right] (12x^3) \right]
\end{align*}
\]

b) \( g(\theta) = \sqrt{\frac{\theta^2 - \sec(2\theta + \pi)}{\log_{23}(\theta \csc \theta)}} \) 

\[
\begin{align*}
  g'(\theta) &= \frac{1}{2} \left( \frac{\theta^2 - \sec(2\theta + \pi)}{\log_{23}(\theta \csc \theta)} \right)^{-\frac{1}{2}} \frac{d}{d\theta} \left( \theta^2 - \sec(2\theta + \pi) \right) \\
  &= \frac{1}{2} \left( \frac{\theta^2 - \sec(2\theta + \pi)}{\log_{23}(\theta \csc \theta)} \right)^{-\frac{1}{2}} \left( \frac{\theta^2 - \sec(2\theta + \pi) + \tan(2\theta + \pi) \tan(\theta + \pi)}{\log_{23}(\theta \csc \theta)} \right) \\
  &= \frac{1}{2} \left( \frac{\theta^2 - \sec(2\theta + \pi) + \tan(2\theta + \pi) \tan(\theta + \pi)}{\log_{23}(\theta \csc \theta)} \right) \\
  &= \left( \frac{\theta^2 - \sec(2\theta + \pi) + \tan(2\theta + \pi) \tan(\theta + \pi)}{\log_{23}(\theta \csc \theta)} \right)^2
\end{align*}
\]

6. Given \( k(p) = \frac{5}{p} \) and \( p(h) = 1 + 4e^{0.6h} \), find \( \frac{dk}{dh} \) when \( h = 1 \).

\[
\begin{align*}
  \frac{dk}{dh} &= \frac{dk}{dp} \cdot \frac{dp}{dh} \Rightarrow \frac{p(0) - 5(1)}{p^2} = -\frac{5}{p^2} = \frac{dk}{dp} \text{ and} \\
  4 \cdot 1.6^h (0.6) &= \frac{dp}{dh} \Rightarrow \frac{dk}{dh} = \frac{-5}{(1+4e^{0.6h})^2} \cdot (4e^{0.6h}(0.6)) \\
  \text{and when } h = 1 \Rightarrow \frac{dk}{dh} &= \frac{-3.183}{4} \\
  \text{substitute } p = 1 + 4e^{0.6h}
\end{align*}
\]
7. The position of a particle is given by \( s = f(t) = t^3 - 15t^2 + 48t, t \geq 0 \), where \( s \) is measured in meters and \( t \) in seconds.

a) When is the particle moving backward? \( V(t) < 0 \) \( \Rightarrow \) \( V(t) = 3t^2 - 30t + 48 = 3(t^2 - 10t + 16) = 3(t - 4)(t - 4) \) m/s.

\[ 2 < t < 8 \text{ Sec.} \]

b) Find the acceleration at \( t = 4 \). Is the particle at rest, speeding up, or slowing down at this time?

\[ a(t) = V'(t) = 6t - 30 \Rightarrow a(4) = -6 \text{ m/s}^2 \]

Since \( V(4) < 0 \) and \( a(4) < 0 \) (i.e., vel. + accel. have same sign) \( \Rightarrow \) **speeding up**

8. Use the information in the table below regarding the functions \( f(x) \) and \( g(x) \) to answer questions a) - c).

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>5</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>1</td>
<td>-3</td>
<td>-5</td>
<td>-8</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>-2</td>
<td>-1</td>
<td>-4</td>
<td>-6</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>2</td>
<td>7</td>
<td>12</td>
<td>18</td>
</tr>
<tr>
<td>( g'(x) )</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>-2</td>
</tr>
</tbody>
</table>

\* Note: \( g(64) = 2 \) 
\* Note: \( g'(64) = -3 \)

a) If \( h(x) = x^2 - 3(g(x))^4 \), find \( h'(5) \).

\( h'(x) = 2x - 12(g(x))^3 g'(x) \) \( \Rightarrow \) \( h'(5) = 2(5) - 12(g(5))^3 g'(5) \)

\( = 10 - 12(7)^3(4) = 10 - 16454 \)

b) If \( j(x) = f(x)g(x^2) \), find \( j'(8) \).

\( j'(x) = f(x)g'(x^2)[2x] + g(x^2)F'(x) \) \( \Rightarrow \) \( j'(8) = f(8)g'(64)16 \)

\( = -5(-3)(16) + 2(-4) = 832 \)

\( q(64)f'(8) \)

\( c) \) If \( k(x) = f(f(g(x))) \), find \( k'(0) \).

\( k'(x) = f'(f(g(x)))[f'(g(x))g'(x)] \) \( \Rightarrow \) \( k'(0) = f'(f(2))[f'(2)g'(0)] \)

\( q(0) = 2 \) \( g(0) = 2 \)

\( = f'(10)[1 \cdot 3] = -6(1 \cdot 3) = -18 \)
9. Sketch the graph of a function that satisfies all of the given conditions.

- \( f(-5) = 8 \) and \( f(2) = -2 \)
- \( f''(2) = f''(4) = 0 \) tangent at \( x = 2 \) and \( x = 4 \)
- \( f'(x) > 0 \) on \((-\infty, -5), (2, 4)\) increasing
- \( f'(x) < 0 \) on \((-5, 0), (0, 2), (4, \infty)\) decreasing
- \( f''(x) > 0 \) on \((-\infty, -2), (0, 3)\) concave up
- \( f''(x) < 0 \) on \((-2, 0), (3, \infty)\) concave down
- vertical asymptote \( x = 0 \)
- \( \lim_{x \to -\infty} f(x) = 1 \) (harsh, asymptotic)

- Note: \( f'(x) \) undefined at \( x = -5 \)

10. Find the equation of the line tangent to the curve \( y = x^4 + \sqrt{3}x^2 + \frac{x^2}{\sqrt{x^6 - 5x^3}} \) at \( x = 2 \).

\[
\begin{align*}
\frac{dy}{dx} &= x^4 + (3x^4)^\frac{1}{2} + \frac{x^2}{(x^6 - 5x^3)^\frac{1}{2}} \\
\frac{dy}{dx} &= 4x^3 + \frac{1}{4}(3x^4)^\frac{-1}{2}(12x^2) + \left[ \frac{(x^6 - 5x^3)^\frac{1}{2}(2x) - x^2 \left[ \frac{1}{2}(x^6 - 5x^3)^\frac{-1}{2}(6x^5 - 15x^2) \right]}{(x^6 - 5x^3)^\frac{3}{2}} \right] \\
\frac{dy}{dx}(2) &= 31.7120 \\
y &= f(2) = 19.5557 \\n\Rightarrow y - 19.5557 &= 31.7120(x - 2)
\end{align*}
\]
11. Find the derivative of the function \( f(x) = \frac{x}{x+2} \) using the definition of the derivative.

\[
\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{x+h-x}{(x+h+2)(x+2)} = \lim_{h \to 0} \frac{2h}{h(x+h+2)(x+2)} = \lim_{h \to 0} \frac{2}{(x+2)(x+2)} = \frac{2}{(x+2)^2} = f'(x)
\]

12. Find the linearization (i.e., linear approximation) of the function \( g(x) = \sqrt[3]{3-x} \) at \( a = -29 \) and use it to approximate \( \sqrt[3]{30.15} \). Is your approximation an overestimate or an underestimate?

\( L(x) = f(a) + f'(a)(x-a) \)  
\( \Rightarrow L(x) = f(-29) + f'(-29)(x+29) \)  
\( f(-29) = \sqrt[3]{3-(-29)} = 2 \)  
\( f'(-29) = \frac{1}{3}(3-x)^{-4/3}(-1) = f'(-29) = -0.025 \)  
\( \Rightarrow L(x) = 2 - 0.025(x+29) \)  
\( \text{or} \ L(x) = -0.025x + 1.6375 \)  
\( \sqrt[3]{30.15} \Rightarrow 3-x = 30.15 \Rightarrow x = -27.15 \Rightarrow L(-27.15) = 1.9769 \)

\( \sqrt[3]{30.15} \Rightarrow 3-x = 30.15 \Rightarrow x = -27.15 \Rightarrow L(-27.15) = 1.9769 \)

* Graph \( g(x) \) and \( L(x) \) and see that \( L(x) \) lies above \( g(x) \) at \( x = -27.15 \) \Rightarrow approximation is overestimate

* Note: Calculator gives 1.9763.

13. Use calculus to determine where \( f(x) = \frac{\ln x}{x} \) is decreasing.

\( f'(x) = \frac{x(\frac{1}{x}) - \ln(x)(1)}{x^2} = \frac{1 - \ln x}{x^2} < 0 \), Since \( x^2 \) is always positive in denominator, \( 1 - \ln x < 0 \) for \( f'(x) < 0 \Rightarrow 1 < \ln x \Rightarrow e^1 < e^{\ln x} \Rightarrow e < x \Rightarrow (e, \infty) \)

* Note: Increasing on \((0, e)\)
14. Let \( y = 3x^4 \sin^2(\pi x) \).

a) Find the differential \( dy \).

\[
dy = f'(x) \, dx = \left[ 3x^4 \left( a(\sin(\pi x))(\cos(\pi x)) \pi + \sin^3(\pi x) \right) \right] \, dx
\]

b) Evaluate \( dy \) and \( \Delta y \) if \( x = 1/2 \) and \( dx = \Delta x = 0.2 \).

Since \( \cos(\pi \cdot 1/2) = 0 \),

\[
dy = \sin^3(\pi \cdot 1/2) \left( 12 \cdot (1/2)^3 \right)(0.2) = (1)(12 \cdot 1/8)(0.2) = 0.3
\]

\[
\Delta y = f(x+\Delta x) - f(x) = f(0.7) - f(0.5)
\]

\[
f(0.7) - f(0.5) = 0.2839
\]

c) Sketch a diagram to illustrate the line segments with lengths \( dx \), \( dy \), and \( \Delta y \).