Section 1.2: Dot Product

**Definition:** Recall that the Work $W$ done by a constant force $F$ in moving an object through a distance $d$ is $W = Fd$. However, this formula only applies when the force is applied in the direction of motion. Suppose now we have an object moving from the point $P$ to the point $Q$ under a force $F$ as shown.

![Diagram showing vectors $\vec{F}$ and $\vec{D}$]

Then the work $W$ done in moving the object from $P$ to $Q$ depends on two things:
(a) The distance the object has moved, namely $|\vec{D}| = |\vec{PQ}|$
(b) The magnitude of the force applied in the direction of motion, that is $|\vec{PS}| = |\vec{F}| \cos(\theta)$. Thus the work $W$ done in moving the object is given by

$$W = |\vec{F}| |\vec{D}| \cos(\theta)$$

**EXAMPLE 1:** Find the work done by a force of 20 lbs acting in the direction $N50^\circ W$ in moving an object 4 feet due west.

![Diagram showing vectors $\vec{F}$ and $\vec{W}$]

$$W = |\vec{F}| \cos(\theta) = (20 \text{ lbs})(4 \text{ feet}) \cos 40^\circ$$

$$= 80 \cos 40^\circ \text{ ft-lbs} \\ \\ \approx 61.3 \text{ ft-lbs}$$

**Definition:** The dot product of the vectors $\vec{a}$ and $\vec{b}$ is defined as $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos(\theta)$, where $\theta$ is the angle between $\vec{a}$ and $\vec{b}$.

**EXAMPLE 2:** Find the dot product of the vectors $\vec{a}$ and $\vec{b}$ if it is known that $\vec{a}$ is a unit vector, $|\vec{b}| = 5$ and $\theta = 30^\circ$.

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos(\theta)$$

$$= (1)(5) \cos(30^\circ)$$

$$= 5 \cdot \frac{\sqrt{3}}{2}$$

Title : Sep 3-11:15 AM (Page 1 of 8)
**Theorem:** \( \vec{a} \cdot \vec{b} = \langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1b_1 + a_2b_2 \). This can be proved using the Law of Cosines, which can be found on page 56 of the Stewart Text.

Note:

(a) \( \vec{a} \) is perpendicular to \( \vec{b} \) if \( \theta = 90^\circ \). How can we tell if two vectors are perpendicular?

\[
\vec{a} \cdot \vec{b} = \langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1b_1 + a_2b_2 = 0
\]

(b) \( \vec{a} \) is parallel to \( \vec{b} \) if \( \theta = 0^\circ \) or \( \theta = 180^\circ \). How can we tell if two vectors are parallel?

\[
\vec{a} = \langle a_1, a_2 \rangle \quad \text{and} \quad \vec{b} = \langle b_1, b_2 \rangle
\]

**Example 3:**

a.) Find the value(s) of \( x \) for which the vectors \( \langle x, 5x \rangle \) and \( \langle x, -10 \rangle \) are perpendicular.

\[
\langle x, 5x \rangle \cdot \langle x, -10 \rangle = 0
\]

Solve:

\[
\begin{align*}
2x^2 - 50x &= 0 \\
x(x - 50) &= 0
\end{align*}
\]

\( x = 0 \) or \( x = 50 \)

b.) Find the value(s) of \( x \) for which the vectors \( \langle 2, x \rangle \) and \( \langle x - 1, 3 \rangle \) are parallel.

"Slope" of \( \langle 2, x \rangle = \frac{x}{2} \)

"Slope" of \( \langle x - 1, 3 \rangle = \frac{3}{x-1} \)

Parallel vectors have the same slope.

\[
\frac{x}{2} = \frac{3}{x-1}
\]

\( x(x-1) = 6 \)

\( x^2 - 6x - 6 = 0 \)

\( (x-3)(x+2) = 0 \)

\( x = 3, \ x = -2 \)
EXAMPLE 4: Find the dot product of the vectors $4\vec{i} + \vec{j}$ and $-3\vec{j}$.

\[
\begin{align*}
4\vec{i} &= 4\langle 1, 0 \rangle = \langle 4, 0 \rangle \\
\vec{j} &= \langle 0, 1 \rangle \\
4\vec{i} + \vec{j} &= \langle 4, 0 \rangle + \langle 0, 1 \rangle = \langle 4, 1 \rangle \\
-3\vec{j} &= \langle 0, -3 \rangle
\end{align*}
\]

\[
\text{dot product: } \langle 4, 1 \rangle \cdot \langle 0, -3 \rangle = 4(0) + 1(-3) = -3
\]

EXAMPLE 5: Using the formula $\vec{a} \cdot \vec{b} = ||\vec{a}|| ||\vec{b}|| \cos(\theta)$, find the angle between the vectors $\langle 1, 5 \rangle$ and $\langle -2, 3 \rangle$.

\[
\cos \theta = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}|| ||\vec{b}||}
\]

\[
\begin{align*}
\cos \theta &= \frac{\langle 1, 5 \rangle \cdot \langle -2, 3 \rangle}{||\langle 1, 5 \rangle|| ||\langle -2, 3 \rangle||} \\
&= \frac{-2 + 15}{\sqrt{1^2 + 5^2} \sqrt{(-2)^2 + 3^2}} \\
&= \frac{13}{\sqrt{13 \cdot 13}} \\
&= \frac{13}{\sqrt{169}} \\
&= \frac{13}{13} \\
&= 1
\end{align*}
\]

\[\theta = 45^\circ\]

EXAMPLE 6: The points $A(2, -2)$, $B(1, 4)$ and $C(0, -6)$ form a triangle. Find the angle, $\alpha$, located at the vertex $A$.

\[
\begin{align*}
\vec{AB} &= \langle 1, 4 \rangle \\
\vec{AC} &= \langle -2, -4 \rangle
\end{align*}
\]

\[
\begin{align*}
\cos \alpha &= \frac{\vec{AB} \cdot \vec{AC}}{||\vec{AB}|| ||\vec{AC}||} \\
&= \frac{\langle 1, 4 \rangle \cdot \langle -2, -4 \rangle}{\sqrt{1^2 + 4^2} \sqrt{(-2)^2 + (-4)^2}} \\
&= \frac{-2 - 16}{\sqrt{17} \sqrt{20}} \\
&= \frac{-18}{\sqrt{340}} \\
&\Rightarrow \cos \alpha = \frac{-18}{\sqrt{340}} \approx 0.7071 \\
\end{align*}
\]

\[\alpha \approx 144^\circ\]
EXAMPLE 7: A force with representation \( \vec{F} = (3, 8) \) moves an object along a straight line from the point \((2, 3)\) to the point \((4, 5)\). Find the work done if the distance is measured in meters and the magnitude of the force is measured in Newtons.

\[
\text{displacement vector } \vec{D} = \langle 2, 2 \rangle
\]

\[
W = |F| \cdot |D| \cos \theta \quad \text{if we were given } \theta
\]

\[W = \vec{F} \cdot \vec{D} \quad \text{if given components of } \vec{F} \text{ and } \vec{D}.
\]

\[
\vec{F} = \langle 3, 8 \rangle \quad W = \vec{F} \cdot \vec{D}
\]
\[
\vec{D} = \langle 2, 2 \rangle = \langle 3, 8 \rangle - \langle 2, 2 \rangle
\]
\[
= (6 + 16) = 22 \text{ Nm}
\]
\[
\approx 22 \text{ J}
\]

EXAMPLE 8: A woman exerts a horizontal force of 65 lb on a crate as she pushes it up a ramp that is 20 ft long and inclined at an angle of 20° above the horizontal. Find the work done on the box.

\[
W = |F| \cdot |D| \cos \theta
\]

\[
= (65 \text{ lbs})(20 \text{ ft}) \cos 20°
\]
\[
\approx 1221.6 \text{ ft lbs}
\]
**Definition:** If \( \vec{a} = (a_1, a_2) \), then the orthogonal complement of \( \vec{a} \) is \( \vec{a}^\perp = (-a_2, a_1) \). \( \vec{a}^\perp \) is perpendicular to \( \vec{a} \) and has the same length as \( \vec{a} \). Note: There is a second vector that is orthogonal to \( \vec{a} \), namely \( (a_2, -a_1) \), it just does not have a special name.

**Example 9:** Find the orthogonal complement of \( \vec{a} = (-1, 4) \). Graph both \( \vec{a} \) and \( \vec{a}^\perp \) on the same axis.

\[
\vec{a} = \langle -1, 4 \rangle \quad m = \frac{4}{-1} = -4 \\
\vec{a}^\perp = \langle -4, -1 \rangle \quad m = \frac{-1}{-4} = \frac{1}{4}
\]

**Example 10:** Find two unit vectors perpendicular to \( (2, -3) \).

1. \( \vec{a} = \langle 2, -3 \rangle \) \( \quad \vec{u} = \frac{\langle 3, 2 \rangle}{\sqrt{13}} = \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \rangle \)

2. \( \vec{a} = \langle -3, -2 \rangle \) \( \quad \vec{u} = \frac{\langle -3, -2 \rangle}{\sqrt{13}} = \langle \frac{-3}{\sqrt{13}}, \frac{-2}{\sqrt{13}} \rangle \)
Vector and Scalar Projections: Given $a = \langle a_1, a_2 \rangle$ and $b = \langle b_1, b_2 \rangle$, we want to project $b$ onto $a$.

- The Scalar Projection of $b$ onto $a$ (also called the component of $b$ onto $a$) is:
  \[ \text{comp}_a b = \frac{a \cdot b}{|a|} \]

To project $\hat{b}$ onto $\hat{a}$, drop a perpendicular from $b$ onto $a$.

- The Vector Projection of $b$ onto $a$ is:
  \[ \text{proj}_a b = \frac{a \cdot b}{|a|^2} a \]

\[ \text{proj}_a b = \left( \frac{a \cdot b}{|a|} \right) \left( \frac{\hat{a}}{|\hat{a}|} \right) \]
EXAMPLE 11: Find the vector and scalar projection of \( \langle 4, 8 \rangle \) onto \( \langle 2, 1 \rangle \).

\[
\text{scalar projection of } \vec{b} \text{ onto } \vec{a} \text{ is} \\
\text{comp}_a \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{\langle 4, 8 \rangle \cdot \langle 2, 1 \rangle}{|\langle 2, 1 \rangle|} \\
= \frac{16}{\sqrt{5}}
\]

vector projection: \[ \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \frac{\vec{a}}{|\vec{a}|} \]

\[ \frac{16}{\sqrt{5}} \frac{\langle 2, 1 \rangle}{\sqrt{5}} = \frac{16}{5} \langle 2, 1 \rangle = \langle \frac{32}{5}, \frac{16}{5} \rangle \]

EXAMPLE 12: Find the length of the vector projection of \( \langle 2, 1 \rangle \) onto \( \langle -5, 1 \rangle \).
EXAMPLE 13: Find the distance from the point $P(2, 1)$ to the line $y = 2x + 1$. 