Instructions  Please write your solutions on your own paper. Explain your reasoning in complete sentences to maximize credit.

1. Suppose $f$ is a function such that $f(0) = 1$ and $f'(0) = 3$. Let $g$ be the composite function such that $g(x) = f(\sin x)$. Determine the value of the derivative $g'(0)$.

**Solution.** By the chain rule, $g'(x) = f'(\sin x) \cos x$. Therefore $g'(0) = f'(\sin 0) \cos 0 = f'(0) = 3$.

2. Suppose $f$ is a function such that $f(2) = 4$ and $f'(2) = 7$. Let $g$ be the composite function such that $g(x) = e^{f(2x)}$. Determine the value of the derivative $g'(1)$.

**Solution.** By the chain rule, $g'(x) = e^{f(2x)} f'(2x) \times 2$. Therefore $g'(1) = 2e^{f(2)} f'(2) = 14e^4$ (which is approximately equal to 764.37).

3. Suppose $f$ and $g$ are inverse functions [recall this means that $f(g(x)) = x$ and $g(f(x)) = x$], and suppose $f(1) = 2$, $f'(1) = 3$, $f(2) = 5$, and $f'(2) = 7$. Determine the value of the derivative $g'(2)$.

**Solution.** According to the rule for the derivative of an inverse function, $g'(2) = 1/f'(1) = 1/3$.

You could also work out the answer from the chain rule, starting from $g(f(x)) = x$. Differentiating gives $g'(f(x)) f'(x) = 1$, so $g'(f(1)) f'(1) = 1$, or $g'(2) = 1/f'(1) = 1/3$.

4. According to the TI-89 calculator, the functions $\ln(x)$ and $\ln(171x)$ have the same derivative: namely, $1/x$. Does it make sense that the graphs of these two different functions have the same slope? Explain.

**Solution.** One of the basic properties of logarithms tells us that $\ln(171x) = \ln(171) + \ln(x)$. Thus the graph of $\ln(171x)$ is the same as the graph of $\ln(x)$ except translated upward by the fixed amount $\ln(171)$. Translating the tangent lines upward does not change their slope. To put it another way, the two functions have the same derivative because they differ by a constant (and the derivative of a constant function is 0).
5. Suppose \( f(2) = 3 \) and \( f'(2) = 5 \). Let \( g \) be the function such that \( g(x) = x^f(x) \). Use logarithmic differentiation to determine the value of the derivative \( g'(2) \).

**Solution.** Since \( \ln g(x) = f(x) \ln x \), differentiating (using the chain rule on the left-hand side and the product rule on the right-hand side) shows that \( g'(x)/g(x) = f'(x) \ln x + f(x)/x \). Consequently, \( g'(2)/g(2) = f'(2) \ln 2 + f(2)/2 \), or \( g'(2) = 2^2(5 \ln 2 + 3/2) = 40 \ln 2 + 12 \) (which is approximately equal to 39.7).

6. Show that the curves \( y = x^3 \) and \( x^2 + 3y^2 = 1 \) intersect orthogonally.

**Solution.** For the first curve, the slope is \( y' = 3x^2 = 3y/x \). For the second curve, implicit differentiation shows that \( 2x + 6yy' = 0 \), or \( y' = -x/(3y) \). The product of the two slopes is equal to \(-1\), which means that the curves intersect at right angles.

It is not necessary to know the intersection points, which are approximately \( \pm (0.7325, 0.393) \).

7. Two students leave Milner Hall at the same time, walking in perpendicular directions. One student walks northeast along Ross Street at a speed of 3 feet per second, and the other student walks northwest along Asbury Street at a speed of 4 feet per second. At what rate is the distance between the students increasing 10 seconds after they start walking?

**Solution.** Let \( x(t) \) denote the distance that the first student has walked by time \( t \), let \( y(t) \) denote the distance that the second student has walked by time \( t \), and let \( D(t) \) denote the distance between the students at time \( t \). We are given that \( dx/dt = 3 \) and \( dy/dt = 4 \), and we are supposed to determine \( dD/dt \) when \( t = 10 \). The relation between the variables is \( D^2 = x^2 + y^2 \) (by the Pythagorean law for right triangles).

Differentiating shows that \( 2D(dD/dt) = 2x(dx/dt) + 2y(dy/dt) \), or \( dD/dt = (3x + 4y)/D \). Now \( x = 3t \) and \( y = 4t \), so \( D = 5t \). Therefore \( dD/dt = (9t + 16t)/(5t) = 5 \), so the rate of change of the distance \( D \) with respect to time is 5 feet per second. This rate is independent of \( t \), so we do not need to use the information that \( t = 10 \).
8. Consider the curve given in parametric form by $x(t) = t^3$ and $y(t) = \sin(t)$, where $t$ runs over the real numbers. Do these parametric equations determine $y$ as a one-to-one function of $x$? Explain why or why not.

**Solution.** The parametric equations determine $y$ as a function of $x$, but it is not a one-to-one function. For example, if $t = 0$ then $x = 0$ and $y = 0$, while if $t = \pi$ then $x = \pi^3$ and $y = 0$. Since the same value of $y$ corresponds to two values of $x$, the function is not one-to-one.

9. If you invest $1,000 at 5% interest compounded continuously, then the amount $A(t)$ in your account after $t$ years is $A(t) = 1,000e^{t/20}$. How many years does it take to double your money?

**Solution.** We need to solve the equation $2,000 = 1,000e^{t/20}$ for $t$. Dividing by 1,000 shows that $2 = e^{t/20}$, so $\ln 2 = \frac{t}{20}$, or $t = 20\ln 2$ (which is approximately 13.86 years).

10. **Optional problem for extra credit**
Jackie finds an approximate value for $\sqrt[3]{1001}$ (the cube root of 1001) by using the linear approximation for the function $\sqrt[3]{x}$ with the base point $a = 1000$. Jamie finds an approximate value for $\sqrt[3]{1001}$ by doing one iteration of Newton’s method applied to the function $x^3 - 1001$ with starting point $x_0 = 10$. Whose approximation to $\sqrt[3]{1001}$ is more accurate?

**Solution.** Jackie’s calculation, using $f(x) = \sqrt[3]{x}$ and $f'(x) = 1/(3x^{2/3})$ and the linear approximation formula $f(x) \approx f(a) + f'(a)(x - a)$, is

$$\sqrt[3]{1001} \approx \sqrt[3]{1000} + \frac{1}{3(1000)^{2/3}}(1001 - 1000) = 10 + \frac{1}{300}.$$ 

Jamie’s calculation, using $f(x) = x^3 - 1001$ and $f'(x) = 3x^2$ and Newton’s formula $x_1 = x_0 - f(x_0)/f'(x_0)$, is

$$x_1 = 10 - \frac{1000 - 1001}{3(10^2)} = 10 + \frac{1}{300}.$$ 

The two approximations are equal. (The equality is no coincidence, for both methods use a tangent line approximation: one for a function and one for the inverse function.)