Instructions Please write your solutions on your own paper. Explain your reasoning in complete sentences to maximize credit.

1. Suppose vector $\vec{v} = 2\vec{i} + 3\vec{j}$ and vector $\vec{w} = a\vec{i} + b\vec{j}$ for some numbers $a$ and $b$. If the length $|\vec{w}| = 53$, and the scalar projection of $\vec{w}$ onto $\vec{v}$ equals 45, find the scalar projection of $\vec{w}$ onto $\vec{v}^\perp$, the orthogonal complement of $\vec{v}$.

   [Recall that the scalar projection of $\vec{w}$ onto $\vec{v}$ is the component of $\vec{w}$ in the direction of $\vec{v}$, that is, the magnitude of the vector projection of $\vec{w}$ onto $\vec{v}$. In two-dimensional space, the orthogonal complement of a vector is the vector obtained by a 90° counterclockwise rotation.]

2. Recall that the floor function $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$. (The textbook denotes this function by the symbol $[x]$ and calls it the greatest integer function.) Use the Squeeze Theorem to prove that

$$\lim_{x \to 0} \left( x^2 \left\lfloor \frac{1}{x^2} \right\rfloor \right) = 1.$$  

   [Neither the TI-89 calculator nor Maple can do this limit!]

3. In February 2006, students at Michigan Tech rolled a world-record snowball, a sphere 6.75 feet in diameter. If the snowball subsequently melted at a rate of 1 cubic foot per day, how fast was the radius of the snowball changing, in units of inches per day, when the radius was equal to 20 inches?

   [One foot equals 12 inches, so one cubic foot equals 1728 cubic inches. The volume of a sphere of radius $r$ equals $\frac{4}{3}\pi r^3$.]

4. The equation $e^y + \ln(x) = 0$ implicitly determines $y$ as a function $f(x)$ when $0 < x < 1$. Show that the graph of $f$ has an inflection point where $x = 1/e$.

5. For how many values of $x$ does the slope of the graph of $\tan^{-1}(x)$ equal the slope of the graph of $1/\tan(x)$? Explain how you know.

   [Recall that $\tan^{-1}(x)$ and $1/\tan(x)$ are different functions, because the exponent $-1$ means inverse function, not reciprocal.]

6. Determine a number $c$ and a function $f$ such that $3 \int_0^x f(t) \, dt = e^{3x} - c$. 

December 11, 2007 Page 1 of 2 Dr. Boas
7. Suppose that a particle moves in a circle and has position vector equal to \((\cos t, \sin t)\) at time \(t\). Show that the corresponding velocity vector is orthogonal to the acceleration vector for every \(t\).
   [This is a standard property of circular motion that you may have learned in physics class.]

8. Give a careful statement of one of the following definitions:
   
   (a) the precise definition of limit (using \(\epsilon\) and \(\delta\));
   
   (b) the definition of derivative (as a limit);
   
   (c) the definition of the definite integral (as a limit of Riemann sums).

9. Sketch an example of a graph having no points of discontinuity, exactly one critical point, and exactly two inflection points.

10. Optional problem for extra credit

    Abby, Blake, and Cory are trying to compute the indefinite integral \(\int 2 \sin(x) \cos(x) \, dx\).

    Abby says, “Substituting \(u = \sin(x)\) and \(du = \cos(x) \, dx\), I get
    \[
    \int 2 \sin(x) \cos(x) \, dx = \int 2u \, du = u^2,
    \]
    so back substituting gives the answer \(\sin^2(x)\).”

    Blake says, “I remember a trig identity \(2 \sin(x) \cos(x) = \sin(2x)\), so substituting \(u = 2x\) and \(du = 2 \, dx\), I get
    \[
    \int 2 \sin(x) \cos(x) \, dx = \int \sin(2x) \, dx = \frac{1}{2} \int \sin(u) \, du = -\frac{1}{2} \cos(u),
    \]
    and back substituting gives the answer \(-\frac{1}{2} \cos(2x)\).”

    Cory says, “I tried this problem on my TI-89 calculator, and it gave me the answer \(-\cos(x)^2\).”

    Who (if anyone) is right, who is wrong, and what mistakes did these students make?