Instructions Please write your name in the upper right-hand corner of the page. Write complete sentences to explain your solutions.

1. Suppose \( f(x) = \frac{\cos x}{2 + \sin x} \). Find the absolute maximum value of this function for \( x \) in the closed interval \([0, 2\pi]\).
   [This is exercise 50 on page 313 of the textbook.]

   \textbf{Solution.} By the quotient rule,
   \[
   f'(x) = \frac{(2 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(2 + \sin x)^2} = \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2 + \sin x)^2}.
   \]
   Since \( \sin^2 x + \cos^2 x = 1 \), the derivative \( f'(x) \) is equal to 0 if and only if \( \sin x = -1/2 \), which means that \( \cos x = \pm\sqrt{3}/2 \). Substituting these values for \( \sin x \) and \( \cos x \) back into the expression for \( f(x) \) shows that at the critical points, the value of the function is \( \pm(\sqrt{3}/2)/(3/2) \), or \( \pm\sqrt{3}/3 \), or \( \pm 1/\sqrt{3} \).

   At the endpoints 0 and \( 2\pi \), the function has the value \( 1/2 \). Since \( 1/2 < 1/\sqrt{3} \), the maximum value of the function is \( 1/\sqrt{3} \) (taken when \( x = 11\pi/6 \)).

2. The graph below shows the derivative \( f'(x) \) on the open interval \((0, 6)\).
   Determine the values of \( x \) for which the graph of the original function \( f(x) \) [not shown] has (a) local minima and (b) inflection points.

   \textbf{Solution.} (a) Local minima occur when \( f(x) \) changes from decreasing to increasing, or when \( f'(x) \) changes from negative to positive, that is,
at $x = 1$ and $x = 5$. (At $x = 3$, there is a local maximum, because the derivative changes from positive to negative.)

(b) There are inflection points when the derivative changes from increasing to decreasing (or vice versa). This happens when $x = 2$ and $x = 4$.

3. Suppose $f$ is a function that has derivatives of all orders. If $f(0) = 0$ and $f'(0) = 2$, compute the limit $\lim_{x\to 0} \frac{f(x) \sin(3x)}{1 - e^{x^2}}$.

**Solution.** *Method 1.* This is a $0/0$ indeterminate form, so by l’Hospital’s rule, the limit equals

$$\lim_{x\to 0} \frac{f'(x) \sin(3x) + 3f(x) \cos(3x)}{-2xe^{x^2}}.$$ 

The new limit is still an indeterminate form, and another application of l’Hospital’s rule gives

$$\lim_{x\to 0} \frac{f''(x) \sin(3x) + 3f'(x) \cos(3x) + 3f(x) \cos(3x) - 9f(x) \sin(3x)}{-2e^{x^2} - 4xe^{x^2}}.$$ 

Substituting $x = 0$ gives the result $-6$.

*Method 2.* A little trickery will reduce the computational complexity. We know that $\lim_{x\to 0} \frac{\sin(3x)}{3x} = 1$, and the given information implies that $\lim_{x\to 0} \frac{f(x)}{2x} = 1$. Therefore

$$\lim_{x\to 0} \frac{f(x) \sin(3x)}{1 - e^{x^2}} = \lim_{x\to 0} \frac{6x^2}{1 - e^{x^2}} = \lim_{t\to 0^+} \frac{6t}{1 - e^t}.$$ 

Now a single application of l’Hospital’s rule produces the answer $-6$.

4. Sketch the graph of a function $f$ that satisfies all of the following conditions.

- Conditions on the function: $f(-1) = 4$ and $f(1) = 0$.
- Conditions on the derivative: $f'(-1) = 0$ and $f'(1)$ does not exist.
- Additional conditions on the derivative: $f'(x) < 0$ if $|x| < 1$ and $f'(x) > 0$ if $|x| > 1$. 


Condition on the second derivative: $f''(x) < 0$ if $x \neq 1$.

[This is exercise 16 on page 306 of the textbook.]

**Solution.** On $(-\infty, -1)$, the graph of the function $f$ is increasing and concave down; on $(-1, 1)$, the graph is decreasing and concave down; there is a local maximum at $x = -1$; and on $(1, \infty)$, the graph is increasing and concave down. Since $f'(1)$ does not exist, either the left-hand derivative at $x = 1$ and the right-hand derivative are unequal, or one of these one-sided derivatives fails to exist. An example of a graph satisfying all these properties is shown below.

![Graph](image-url)

This particular graph corresponds to the following piecewise-defined function:

$$f(x) = \begin{cases} 
4 - (x + 1)^2, & \text{if } x \leq 1, \\
\ln(x), & \text{if } x > 1.
\end{cases}$$