1. Write down a function $f(x)$ whose graph looks like the picture. The key features of the picture are that $\lim_{x \to 1^+} f(x) = \infty$, $\lim_{x \to 1^-} f(x) = -\infty$, $\lim_{x \to -\infty} f(x) = 2$, and $\lim_{x \to \infty} f(x) = 2$. Explain the reasoning for your choice of $f(x)$.

**Solution.** One way to get a suitable function is to start with the simplest function you know that has both horizontal and vertical asymptotes and then adjust the function.

The function $1/x$ has the coordinate axes as asymptotes. Shifting the function to $1/(x - 1)$ adjusts the vertical asymptote to the required place. Further shifting the function to $2 + \frac{1}{x - 1}$ gives the required horizontal asymptote. You could simply the expression to $\frac{2x - 1}{x - 1}$.

There are other functions that have the same asymptotes, but this one is the simplest that matches the $x$ and $y$ intercepts.
2. The TI-89 calculator says that $\lim_{x \to 1} \left( \frac{1}{x - 1} - \frac{2}{x^2 - 1} \right) = \frac{1}{2}$. Supply a computation that confirms this value. (Suggestion: combine the fractions with a common denominator and simplify.)

**Solution.** When $x \neq 1$, we can simplify the expression algebraically as follows:

$$
\frac{1}{x - 1} - \frac{2}{x^2 - 1} = \frac{x - 1}{x^2 - 1} - \frac{1}{x + 1}.
$$

The expression on the right-hand side is a rational function whose denominator is not equal to 0 when $x = 1$, so the function is continuous at $x = 1$. Therefore, we can correctly obtain the limit by substituting in the value of $x$: namely,

$$
\lim_{x \to 1} \left( \frac{1}{x - 1} - \frac{2}{x^2 - 1} \right) = \lim_{x \to 1} \frac{1}{x + 1} = \frac{1}{2}.
$$

3. Find a number $c$ such that $\lim_{x \to \infty} \left( \sqrt{x^2 + cx} - x \right) = 3$.

**Solution.** We make some algebraic manipulations to convert the expression into a more manageable form. Multiplying and dividing by $\sqrt{x^2 + cx} + x$ gives the equivalent expression

$$
\frac{x^2 + cx - x^2}{\sqrt{x^2 + cx} + x} = \frac{c}{\sqrt{1 + \frac{c}{x}} + 1}.
$$

Now $\lim_{x \to \infty} \frac{c}{x} = 0$, so the limit of the whole expression is equal to $\frac{c}{\sqrt{1+0} + 1}$ or $c/2$. In order for this result to equal 3, we must have $c = 6$. 