Comments on reading and writing proofs

A student wrote the following.

**Theorem 1.** If $x < y$, then $x^2 < y^2$.

**Proof.** Multiplying by $x$, $x < y \Rightarrow x \cdot x < y \cdot x$. Multiplying by $y$, $x < y \Rightarrow x \cdot y < y \cdot y$. But by commutativity of multiplication, $y \cdot x = x \cdot y$. By transitivity of $<$, $x \cdot x < y \cdot y$. QED

The student had the right idea, but the presentation has some mistakes. The most serious error is that the theorem is false. For example, if $x = -5$ and $y = 1$, then $x < y$, but $x^2$ is not less than $y^2$. The theorem would be valid, however, under the additional hypothesis that $x$ is positive.

Some less serious errors are mistakes in style. For example, it is confusing to separate different formulas by just a comma; there ought to be words between formulas to help the reader see where one formula starts and the other ends. Also, symbols like $\Rightarrow$ and $<$ belong in formulas, but not in the middle of sentences as abbreviations for words.

The introductory phrase “Multiplying by $x$” is a grammatical error known as a “dangling participle”. Who or what is doing the multiplying?

Here is a revision of the student’s work. Notice that the revised exposition consists mostly of words, with only a few formulas.

**Theorem 2.** If $x$ and $y$ are positive real numbers, and $x < y$, then $x^2 < y^2$.

**Proof.** It is valid to multiply both sides of an inequality by a positive real number. Apply this principle twice to the hypothesis that $x < y$. Multiply first by $x$ and then by $y$ to deduce that

$$x^2 < yx \quad \text{and} \quad xy < y^2.$$ 

The commutative law for multiplication says that $yx = xy$, so the transitive property of inequality implies the desired conclusion that $x^2 < y^2$. QED