Math 409-502

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Integration
Some vocabulary
- partition
- mesh
- refinement
- upper sum
- lower sum
- Riemann sum
- integrable function

Some theorems
- Bounded monotonic functions are integrable.
- Bounded continuous functions are integrable.
- Integration and differentiation are inverse operations (fundamental theorem of calculus).

Partition and mesh
A partition of a compact interval \([a,b]\) is a subdivision of the interval.

A compact interval \([1, 5]\)

\[\begin{array}{cccc}
1 & 2 & 4 & 5 \\
\end{array}\]

A partition of the interval: division points 1, 2, 4, 5

\[\begin{array}{cccc}
1 & 2 & 4 & 5 \\
\end{array}\]

Symbolic notation for the division points

\[\begin{array}{cccc}
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\
\end{array}\]

The mesh of a partition is the maximum width of the subintervals. In the above example, the mesh is 4 – 2 = 2.
**Upper sum**

The *upper sum* of a bounded function for a partition of a compact interval means the sum over the subintervals of the supremum of the function on the subinterval times the width of the subinterval.

Symbolic notation: \[ \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{[x_{j-1}, x_j]} f(x). \]

**Lower sum**

The *lower sum* is defined similarly with the infimum in place of the supremum.

Symbolic notation: \[ \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f(x). \]
Integrable functions

A function defined on a compact interval \([a, b]\) is integrable if (i) the function is bounded, and (ii) for every \(\epsilon > 0\), there exists \(\delta > 0\) such that for every partition of mesh \(< \delta\) the upper sum for the partition and the lower sum for the partition differ by less than \(\epsilon\).

Example. A constant function is integrable because every upper sum equals every lower sum.

Example. \(f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}\)

is not integrable because every lower sum equals 0, but every upper sum equals the width of the interval.

Homework

- Read sections 18.1 and 18.2, pages 241–244.
- Consider the integrable function \(f(x) = x\) on the interval \([1, 2]\). How small must the mesh of a partition be in order to guarantee that the upper sum and the lower sum differ by less than 1/10?
- Do exercise 18.2/3 on page 248.