Reminders on radius of convergence

We can find the radius of convergence of a power series by either the ratio test or the root test, but some other test is needed to determine the endpoint behavior.

Useful tests for endpoint behavior are:

- $n$th-term test
- comparison tests
- alternating series test

Follow-up on endpoint convergence

Last time we saw (by the ratio test) that \( \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n \) has radius of convergence equal to 4, and \( \sum_{n=1}^{\infty} \frac{n! x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \) has radius of convergence equal to 2. At the right-hand endpoint, both series become \( \sum_{n=1}^{\infty} \frac{(n!)^2 4^n}{(2n)!} \). That series diverges by the $n$th-term test. Indeed,

\[
4^n = (1 + 1)^{2n} = 1 + \binom{2n}{1} + \binom{2n}{2} + \cdots + \binom{2n}{n} + \cdots + \binom{2n}{1} + 1,
\]

so \( 4^n > \binom{2n}{n} = \frac{(2n)!}{(n!)^2} \). Thus \( \frac{(n!)^2 4^n}{(2n)!} > 1 \), so the series cannot converge. For the same reason, divergence occurs at the left-hand endpoint in this example.
Operations on power series

Addition, subtraction, multiplication, and division of power series work the way you expect.

Example

\[
\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \]

\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \]

so the coefficient of \(x^5\) in the product \(\cos(x) \sin(x)\) equals \(\frac{1}{5!} + \frac{1}{2 \cdot 3!} + \frac{1}{4!} = \frac{2}{15}\).

Remark on the multiplication theorem

Theorem (page 121): If \(\sum_{n=0}^{\infty} a_n\) and \(\sum_{n=0}^{\infty} b_n\) both converge absolutely, then the product of the two series equals the absolutely convergent series \(\sum_{n=0}^{\infty} c_n\), where \(c_n = \sum_{k=0}^{n} a_k b_{n-k}\).

Counterexample in case of conditional convergence: Set \(a_n = b_n = (-1)^n / \sqrt{n + 1}\). Then \(\sum a_n\) and \(\sum b_n\) are conditionally convergent by the alternating series test, but the series \(\sum c_n\) is divergent. Indeed, \(c_n = \sum_{k=0}^{n} \frac{(-1)^k (-1)^{n-k}}{\sqrt{k+1} \sqrt{n-k+1}}\). All the terms in this sum have the same sign \((-1)^n\), so \(|c_n| \geq \sum_{k=0}^{n} \frac{1}{n+1} = 1\). Hence \(\sum c_n\) diverges.
Homework

1. Read Chapter 9, pages 125–134.