Math 409-502

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Summary of convergence tests

- If $a_n \not\to 0$, then $\sum_n a_n$ diverges.

- Comparison tests for positive series: if $a_n \leq b_n$ for all large $n$, or alternatively if $a_n/b_n$ has a finite limit, then convergence of $\sum_n b_n$ implies convergence of $\sum_n a_n$.

- Absolute convergence implies convergence: if $\sum_n |a_n|$ converges, then so does $\sum_n a_n$.

- Ratio and root tests: if either $\lim_{n \to \infty} |a_n|^{1/n}$ or $\lim_{n \to \infty} |a_{n+1}/a_n|$ exists and is strictly less than 1, then $\sum_n a_n$ converges.

- Special tests for decreasing positive terms $a_n$:
  (i) if $f(x) \downarrow 0$ as $x \to \infty$, then $\int_\infty^\infty f(x) \, dx$ and $\sum_\infty^\infty f(n)$ have the same convergence/divergence behavior (integral test); (ii) if $a_n \downarrow 0$, then $\sum_n (-1)^n a_n$ converges (alternating series test).

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Cauchy’s condensation test

Another special test for decreasing terms

Suppose $0 < a_{n+1} \leq a_n$ for all (large) $n$. Then the two series $\sum_n a_n$ and $\sum_n 2^n a_{2^n}$ either both converge or both diverge.

**Example:** $\sum_n \frac{1}{n \ln(n)}$

Since $\frac{1}{n \ln(n)}$ is a decreasing function of $n$, the test applies and says that the convergence/divergence behavior is the same as for the series $\sum_n 2^n \frac{1}{2^n \ln(2^n)}$. That simplifies to $\sum_n \frac{1}{n \ln(2)}$, which is a multiple of the divergent harmonic series. Therefore the original series $\sum_n \frac{1}{n \ln(n)}$ diverges too.
Proof of the condensation test (sketch)

\[ a_8 + a_9 + \cdots + a_{15} \leq 8a_8 \leq 2(a_4 + a_5 + a_6 + a_7) \]
\[ a_{16} + a_{17} + \cdots + a_{31} \leq 16a_{16} \leq 2(a_8 + a_9 + \cdots + a_{15}) \]

\[ \vdots \]

Adding such inequalities shows that partial sums of \( \Sigma_n 2^n a_{2n} \) are bounded below by partial sums of \( \Sigma_n a_n \) and are bounded above by twice the partial sums of \( \Sigma_n a_n \). Therefore the two series have the same convergence/divergence behavior.

Power series

Example: \( \sum_{n=1}^{\infty} \frac{x^n}{2^n \sqrt{n}} \)

For which values of \( x \) does that series converge?

[This is Exercise 8.1/1a on page 123.]

Solution:

By the root test, the series converges (absolutely) when

\[ 1 > \lim_{n \to \infty} \left| \frac{x^n}{2^n \sqrt{n}} \right|^{1/n} = \lim_{n \to \infty} \frac{|x|}{2 \sqrt{n}^{1/n}} = \frac{|x|}{2} , \]

that is, when \( |x| < 2 \).

The series diverges when \( |x| > 2 \) by the (proof of the) root test.

A different test is needed to see what happens when \( x = \pm 2 \).
Homework

• Read section 8.1, pages 114–117.
• Do Exercise 7.6/1a,c on page 111.
• Do Exercise 8.1/1g on page 123.