Limit theorems

Sums, products, and quotients

If \( a_n \to L \) and \( b_n \to M \) then \( a_n + b_n \to L + M \);
and \( a_n \cdot b_n \to L \cdot M \);
and if in addition \( L \neq 0 \) then \( b_n / a_n \to M / L \).
[In the third case, \( a_n \neq 0 \) when \( n \) is large, so \( b_n / a_n \) makes sense for \( n \) large.]

Squeeze theorem

If \( a_n \leq b_n \leq c_n \) for all sufficiently large \( n \), and if the sequences \( \{a_n\}_{n=1}^{\infty} \) and \( \{c_n\}_{n=1}^{\infty} \) both converge to the same limit, then the sequence \( \{b_n\}_{n=1}^{\infty} \) converges, and to the same limit.

Limit theorems continued

Location theorems

If \( a_n \to L \) and if \( a_n < M \) for all sufficiently large \( n \), then \( L \leq M \).
The example \( a_n = n / (n + 1) \) shows that we cannot draw the conclusion \( L < M \).

If \( a_n \to L \) and \( L < M \), then \( a_n < M \) for all sufficiently large \( n \).
The example \( a_n = (n + 1) / n \) shows that the hypothesis \( L \leq M \) is insufficient.
**Subsequences**

A non-convergent sequence may have convergent subsequences.

Example:

\[
\frac{1}{4}, \frac{3}{4}, 2, \frac{1}{8}, \frac{7}{8}, 4, \frac{1}{16}, \frac{15}{16}, 8, \frac{1}{32}, \frac{31}{32}, 16, \ldots
\]

If a sequence converges, however, then every subsequence converges to the same limit.

Example: \(\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{13}, \frac{1}{17}, \frac{1}{19}, \ldots\) is a subsequence of the convergent sequence \(\left\{\frac{1}{n}\right\}_{n=1}^{\infty}\), so it converges to 0.

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**Homework**

1. Read sections 5.4 and 5.5, pages 68–73.

2. Do Problem 5-7, page 75.