Instructions Please write your solutions on your own paper. Explain your reasoning in complete sentences. Students in section 500 may substitute problems from part C for problems in part A if they wish.

A Section 500: Do both of these problems.

A.1
In the Euclidean plane $\mathbb{R}^2$, the set of points inside a circle is a disk. Prove that every set of non-overlapping disks in the plane is at most countable. (Non-overlapping means that no two disks intersect.)

A.2
Give an example of a sequence $(a_n)$ of real numbers such that
\[
\inf_n a_n = 1, \quad \liminf_{n \to \infty} a_n = 2, \quad \limsup_{n \to \infty} a_n = 3, \quad \text{and} \quad \sup_n a_n = 4.
\]

B Section 500 and Section 200: Do two of these problems.

B.1
The producer of the television show “The Biggest Loser” proposes to define a metric $d$ on the set of Texas A&M students as follows: $d(x, y) =$ the maximum weight in pounds of students $x$ and $y$ if $x$ and $y$ are different students, and $d(x, y) = 0$ if $x$ and $y$ are the same student. Does this proposed $d$ satisfy all the properties of a metric? Explain.

B.2
Constance misremembers the definition of continuity of a function $f: \mathbb{R} \to \mathbb{R}$ at a point $x$ as the following statement:

For every positive $\epsilon$ and for every positive $\delta$ we have the inequality $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$.

What well-known set of functions does Constance’s property actually characterize? Explain.
B.3
The irrational number $e/\pi$ is approximately equal to 0.865255979432. Does the number $e/\pi$ belong to the Cantor set? Explain how you know.

B.4
Let $(x^{(n)})_{n=1}^\infty$ be a sequence in the space $\ell_2$ of square-summable sequences of real numbers. Thus

$$x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots), \quad \text{and} \quad \|x^{(n)}\|_2 = \left(\sum_{k=1}^\infty |x_k^{(n)}|^2\right)^{1/2}.$$ 

Eleanor conjectures that $x^{(n)} \to 0$ in the normed space $\ell_2$ if and only if $x_k^{(n)} \to 0$ in $\mathbb{R}$ for every $k$. Either prove or disprove Eleanor’s conjecture.

C  Section 200: Do both of these problems.

C.1
For which values of $p$ does the parallelogram law

$$\|x + y\|_p^2 + \|x - y\|_p^2 = 2\|x\|_p^2 + 2\|y\|_p^2$$

hold for all elements $x$ and $y$ in the sequence space $\ell_p$? Explain.

C.2
Alfie proposes the following “proof” that the real numbers between 0 and 1 form a countable set (a statement that we know to be false):

The decimals that have exactly one non-zero digit form a countable set; the decimals that have exactly two non-zero digits form a countable set; and so on. The set of all decimals is therefore the union of countably many countable sets, hence is itself a countable set.

Pinpoint the fatal error in Alfie’s argument.