A Section 500: Do both of these problems.

A.1
Give an example of a metric space that is neither connected, nor totally bounded, nor complete. Say why your example has the required properties.

A.2
Consider the space $C[0, 1]$ of continuous functions on the closed interval $[0, 1]$ provided with the standard metric: $d(f, g) = \max_{0 \leq x \leq 1} |f(x) - g(x)|$. Let $L: C[0, 1] \to \mathbb{R}$ be the function defined via $L(f) = \int_{0}^{1} f(x) \sin(x) \, dx$. Prove that $L$ is a continuous function.

Remark This problem is an instance of the general fact that the Fourier coefficients of a function depend continuously on the function.

B Section 500 and Section 200: Do two of these problems.

B.1
(a) Is it true in every metric space that every closed set is equal to the closure of its interior? Give either a proof or a counterexample.

(b) Is it true in every metric space that every open set is equal to the interior of its closure? Give either a proof or a counterexample.

B.2
Suppose $(M, d)$ is a complete metric space containing at least two points, and suppose there is a point $x_0$ in $M$ such that $(M \setminus \{x_0\}, d)$ is a complete metric space too. Prove that $M$ is disconnected.
B.3
Consider the following two subsets of the real numbers $\mathbb{R}$ equipped with the standard metric: $\mathbb{N}$ is the set of natural numbers 1, 2, 3, ...; and $S$ is the set of reciprocals of natural numbers 1, 1/2, 1/3, .... Show that $S$ is totally bounded, $\mathbb{N}$ is not totally bounded, and $\mathbb{N}$ is homeomorphic to $S$.

Remark This example is a special instance of a general property: namely, a metric space is separable if and only if it is homeomorphic to a totally bounded space.

B.4
In the sequence space $\ell_2$ [the space of sequences $x = (x_1, x_2, ...)$ with norm $\|x\|_2 = (\sum_{n=1}^{\infty} |x_n|^2)^{1/2}$], let $S$ denote the set of absolutely summable sequences [that is, sequences $(x_1, x_2, ...)$ for which the series $\sum_{n=1}^{\infty} |x_n|$ converges]. Prove that $S$ is dense in $\ell_2$ [that is, the closure of $S$ equals the whole space $\ell_2$].

C Section 200: Do both of these problems.

C.1
Suppose $f: (M, d) \to (N, \rho)$ is a function between metric spaces with the property that for every convergent sequence $(x_n)$ in $M$, the image sequence $(f(x_n))$ has a convergent subsequence. Must $f$ be continuous? Supply a proof or a counterexample, as appropriate.

C.2
Connie conjectures that the following statement holds in every complete metric space: If $(F_n)$ is a decreasing sequence of nonempty nested sets (that is, $F_1 \supset F_2 \supset F_3 \supset \cdots$), if the set $F_n$ is both closed and connected for every $n$, and if the intersection $\bigcap_{n=1}^{\infty} F_n$ is nonempty, then the intersection $\bigcap_{n=1}^{\infty} F_n$ must be connected. Either prove Connie’s conjecture or give a counterexample, as appropriate.