The theorem of Mertens about the Cauchy product of infinite series

If the two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge absolutely, then one can freely rearrange terms to find that

$$\left(\sum_{n=0}^{\infty} a_n\right) \left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n,$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. (1)

Franz Carl Joseph Mertens (1840–1927) observed $^1$ that (1) still holds when only one of the first two series, say $\sum_n a_n$, converges absolutely, as long as the second series $\sum_n b_n$ converges conditionally. The argument of Mertens goes as follows.

**Proof.** Let $A_n$, $B_n$, and $C_n$ denote the partial sums $\sum_{k=0}^{n} a_k$, $\sum_{k=0}^{n} b_k$, and $\sum_{k=0}^{n} c_k$. It suffices to prove that both (i) $\lim_{n \to \infty} (C_{2n} - A_n B_n) = 0$ and (ii) $\lim_{n \to \infty} (C_{2n+1} - A_{n+1} B_n) = 0$. For (i), observe that $C_{2n} - A_n B_n$ equals

$$a_0(b_{n+1} + b_{n+2} + \cdots + b_{2n}) + a_1(b_{n+1} + b_{n+2} + \cdots + b_{2n-1}) + \cdots + a_{n-1}b_{n+1} + a_n b_n + b_1 + \cdots + b_{n-1} + a_{n+2}(b_0 + b_1 + \cdots + b_{n-2}) + \cdots + a_{2n} b_0. \quad (2)$$

By hypothesis, there are numbers $A$ and $B$ such that $\sum_{j=0}^{m} |a_j| < A$ and $|\sum_{j=0}^{m} b_j| < B$ for all $m$. Fix a positive $\epsilon$. By hypothesis, there exists a number $N$ such that when $n \geq N$ and $m \geq 1$, one has

$$\sum_{j=n+1}^{n+m} |a_j| < \frac{\epsilon}{A+B} \quad \text{and} \quad \left|\sum_{j=n+1}^{n+m} b_j\right| < \frac{\epsilon}{A+B}.$$

Now (2) shows that when $n \geq N$, one has that $|C_{2n} - A_n B_n| < A \cdot \frac{\epsilon}{A+B} + B \cdot \frac{\epsilon}{A+B} = \epsilon$. Thus $\lim_{n \to \infty} (C_{2n} - A_n B_n) = 0$ as claimed.

To establish the limit (ii), observe that $C_{2n+1} - A_{n+1} B_n$ equals

$$a_0(b_{n+1} + b_{n+2} + \cdots + b_{2n+1}) + a_1(b_{n+1} + b_{n+2} + \cdots + b_{2n}) + \cdots + a_n b_{n+1} + a_{n+2}(b_0 + b_1 + \cdots + b_{n-1}) + a_{n+3}(b_0 + b_1 + \cdots + b_{n-2}) + \cdots + a_{2n+1} b_0,$$

and argue analogously to case (i). $\square$

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