1. Let $E$ denote the open subset of the complex plane defined by $E := \{ z \in \mathbb{C} : |\sin(z)| < |z| \}$. Show that the area of the set $E$ is infinite.

**Solution.** By the triangle inequality, $|\sin z| \leq \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{-y} + e^y) \leq e^{|y|}$. Therefore, if $0 < y < 1$ and $x > 3$, we have $|\sin z| < e < x < |z|$. Hence the set $E$ contains an unbounded half-strip of height 1, so $E$ certainly has infinite area.

2. Solve exercise 2.4 in the textbook: namely, derive the Cauchy-Riemann equations in polar coordinates.

**Solution.** If the derivative $f'$ exists as a two-dimensional limit, then on the one hand $f'(z)$ equals

$$
\lim_{h \to 0} \frac{f((r+h)e^{i\theta}) - f(re^{i\theta})}{he^{i\theta}} = e^{-i\theta} \frac{\partial f}{\partial r}(z),
$$

and on the other hand $f'(z)$ equals

$$
\lim_{\psi \to 0} \frac{f(re^{i(\theta+\psi)}) - f(re^{i\theta})}{re^{i\theta}(e^{i\psi} - 1)} = \frac{e^{-i\theta} \frac{\partial f}{\partial r}(z)}{r} \frac{1}{\partial e^{i\psi}/\partial \psi(0)} = \frac{-ie^{-i\theta}}{r} \frac{\partial f}{\partial \theta}(z).
$$

Equating the two expressions shows that $\frac{\partial f}{\partial r} = \frac{-i}{r} \frac{\partial f}{\partial \theta}$. Taking real and imaginary parts reveals that $\frac{\partial U}{\partial r} = \frac{1}{r} \frac{\partial V}{\partial \theta}$ and $\frac{\partial V}{\partial r} = -\frac{1}{r} \frac{\partial U}{\partial \theta}$.

Alternatively, one can start from the Cauchy-Riemann equations in rectangular coordinates and apply the chain rule:

$$
\frac{\partial U}{\partial r} = \frac{\partial U}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial U}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial U}{\partial x} \cos \theta + \frac{\partial U}{\partial y} \sin \theta,
$$

$$
\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y} \frac{\partial y}{\partial \theta} = -\frac{\partial V}{\partial x} r \sin \theta + \frac{\partial V}{\partial y} r \cos \theta
$$

$$
= \frac{\partial U}{\partial y} r \sin \theta + \frac{\partial U}{\partial x} r \cos \theta.
$$

Hence $\frac{\partial V}{\partial \theta} = r \frac{\partial U}{\partial r}$; and similarly for the second of the Cauchy-Riemann equations.
3. We know that a power series converges absolutely in a certain disk. Consider instead a Dirichlet series of the form
\[ \sum_{n=1}^{\infty} \frac{a_n}{n^z}, \]  
where the complex numbers \( a_n \) are constants (independent of the variable \( z \)), and the expression \( n^z \) means, by definition, \( \exp(z \ln n) \), where \( \ln \) denotes the natural logarithm of a positive real number. Set
\[ A := \limsup_{n \to \infty} \frac{\ln |a_n|}{\ln n}. \]
Supposing that \( A \) is finite, show that the Dirichlet series (1) converges absolutely when \( \text{Re} \ z > A + 1 \).

**Solution.** With no extra work, one can show that the convergence is absolute and uniform in any half-plane where \( \text{Re} \ z \geq B > A + 1 \). Indeed, let \( C \) be a number such that \( B > C > A + 1 \). Then there is a number \( N \) such that \( \frac{\ln |a_n|}{\ln n} < C - 1 \) when \( n > N \). Now \( |a_n| < n^{C-1} \) for such \( n \), which means that
\[ \left| \frac{a_n}{n^z} \right| < \frac{n^{C-1}}{n^B} = \frac{1}{n^{B-C+1}}. \]
Since \( B - C + 1 > 1 \), the series \( \sum_n 1/n^{B-C+1} \) converges, and so the dominated series \( \sum_n a_n/n^z \) converges absolutely and uniformly in the indicated closed half-plane \( \{ z : \text{Re} \ z \geq B \} \).

4. The power series \( 1 - z + z^2 - z^3 + \cdots \) is a geometric series that converges to \( 1/(1 + z) \) when \( |z| < 1 \). Consequently, one expects that the formal anti-derivative
\[ L(z) := z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}z^n \]
should have the properties of a logarithm of \( 1 + z \). Prove that indeed \( \exp(L(z)) = 1 + z \) when \( |z| < 1 \).

You may assume that a power series can be differentiated term by term inside the open disk of convergence (a fact that we have stated but not yet officially proved).
Solution. Let $f(z)$ denote the function $(1 + z)e^{-L(z)}$. Then $f'(z) = e^{-L(z)} - (1 + z)e^{-L(z)}L'(z) = 0$. Hence $f$ is a constant function, so $ce^{L(z)} = 1 + z$. When $z = 0$ one finds that $c = 1$. Hence $e^{L(z)} = 1 + z$.

5. Let $C$ be a continuously differentiable simple closed curve equipped with the standard counterclockwise orientation. Show that $\int_C (\text{Im } z) \, dz$ equals the negative of the area of the region enclosed by the curve $C$.

Solution. Green’s theorem in complex form says that

$$\int_C f(z) \, dz = 2i \int \int \frac{\partial f}{\partial \bar{z}} \, dx \, dy.$$

Since $\text{Im } z = \frac{1}{2i}(z - \bar{z})$, one has that $\frac{\partial \text{Im } z}{\partial \bar{z}} = \frac{-1}{2i}$. Inserting this information into Green’s theorem shows that

$$\int_C \text{Im } z \, dz = \int \int -\, dx \, dy,$$

which indeed equals the negative of the area of the region enclosed by the curve $C$.

6. Suppose $f$ is an analytic function in the unit disk $\{ z \in \mathbb{C} : |z| < 1 \}$. Then, by definition, the function $f$ has a derivative. This problem asks you to show that the function $f$ also is a derivative. Namely, set

$$F(z) := \int_0^1 zf(tz) \, dt \quad \text{when } |z| < 1.$$

Prove that $F$ is differentiable and that $F'(z) = f(z)$.

Solution. Observe that $F(z)$ equals $\int_C f(w) \, dw$, where $C$ is the path parametrized by $tz$, $0 \leq t \leq 1$. Cauchy’s theorem implies, however, that the integral is independent of the path joining 0 to $z$. Consequently, $F(z+h) - F(z)$ equals the integral of $f$ along any path joining $z$ to $z+h$, for instance the path parametrized by $z + th$, $0 \leq t \leq 1$. Therefore

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h} \int_0^1 f(z + th) \, dt = \int_0^1 f(z + th) \, dt.$$

Since $f$ is continuous at $z$, the integral converges when $h \to 0$ to $\int_0^1 f(z) \, dt$, that is, to $f(z)$. Thus $F'(z)$ exists and equals $f(z)$. 

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