An entire function of intermediate growth

Answering Jan Cameron’s question, Dakota Blair suggested the following example of an entire function that grows faster than any polynomial but that has order zero:

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{n!}\right).$$

Since the function has infinitely many zeroes, it is not a polynomial; consequently, as John Paul Ward pointed out, the function must grow faster than any polynomial, by the version of Liouville’s theorem in Exercise 7.8 in the textbook. On the other hand, since $$\sum_{n} (n!)^{-\epsilon}$$ converges for every positive $$\epsilon$$, the function has order zero.$^{1}$

If $$M(r)$$ denotes the maximum of the modulus of our function on a circle of radius $$r$$, then evidently

$$\log M(r) = \sum_{n=1}^{\infty} \log \left(1 + \frac{r}{n!}\right).$$

The preceding general considerations show that $$\log M(r)$$ must grow faster than $$k \log r$$ for every positive constant $$k$$ but slower than $$r^\epsilon$$ for every positive constant $$\epsilon$$. In class, I tried unsuccessfully to show that $$\log M(r)$$ grows like $$(\log r)^2$$, and it turns out that the true growth rate of $$\log M(r)$$ is very slightly slower. In the following proposition, the symbol $$\sim$$ means that the ratio of the two expressions has limit 1.

**Proposition.** We have the asymptotic relation

$$\sum_{n=1}^{\infty} \log \left(1 + \frac{r}{n!}\right) \sim \frac{(\log r)^2}{2 \log \log r} \quad \text{as } r \to \infty.$$

**Proof.** The idea is to break the series at a suitable place and to make different estimates (both from above and from below) on the two sums. To simplify the expression on the right-hand side, it is convenient to replace the variable $$r$$

$^{1}$We did not actually prove in class that the order of a canonical infinite product equals the convergence exponent of the zeroes, but this general property of infinite products follows from a small change in the argument that solves problem 4 on the third take-home examination.
by \( \exp \sqrt{s} \). Then we have to show that

\[
\sum_{n=1}^{\infty} \log \left( 1 + \frac{\exp \sqrt{s}}{n!} \right) \sim \frac{s}{\log s} \quad \text{as } s \to \infty.
\]

Let \( N \) be the unique integer such that

\[
N! \leq \exp \sqrt{s} < (N+1)!
\]

(where \( N \) depends on \( s \)). This integer is a good place to break the series.

Since \( \log(1 + x) < x \) when \( x > 0 \), we have that

\[
0 < \sum_{n=N+1}^{\infty} \log \left( 1 + \frac{\exp \sqrt{s}}{n!} \right) < \sum_{n=N+1}^{\infty} \frac{\exp \sqrt{s}}{n!}.
\]

By the choice of \( N \), all the terms of the series on the right-hand side are less than 1, and each term is less than half the preceding one. Consequently, the series is bounded above by \( \sum_{j=0}^{\infty} 1/2^j \), which converges to 2, a bound that is independent of \( N \) and \( s \). Using Landau’s \( O \) and \( o \) notation (see section 2B of the textbook), we can say that

\[
\sum_{n=1}^{\infty} \log \left( 1 + \frac{\exp \sqrt{s}}{n!} \right) = O(1) + \sum_{n=1}^{N} \log \left( 1 + \frac{\exp \sqrt{s}}{n!} \right).
\]

If \( n \leq N \), then \( 1 + (\exp \sqrt{s})/n! \leq 2(\exp \sqrt{s})/n! \), so

\[
\sum_{n=1}^{N} \log \left( \frac{\exp \sqrt{s}}{n!} \right) < \sum_{n=1}^{N} \log \left( 1 + \frac{\exp \sqrt{s}}{n!} \right) < N \log 2 + \sum_{n=1}^{N} \log \left( \frac{\exp \sqrt{s}}{n!} \right).
\]

Combining this inequality with the preceding equation shows that

\[
\sum_{n=1}^{\infty} \log \left( 1 + \frac{\exp \sqrt{s}}{n!} \right) = O(N) + N \sqrt{s} - \sum_{n=1}^{N} \log(n!).
\]

Notice that the term \( N \sqrt{s} \) cannot be included in the term \( O(N) \), because \( s \) is not a constant (and \( N \) depends on \( s \)).

In estimating the remaining sum, I use that \( \log(n!) = n \log n + O(n) \), which results from the easy part of Stirling’s formula (section 34D in the textbook). Since \( \sum_{n=1}^{N} n = O(N^2) \), we then have that

\[
\sum_{n=1}^{\infty} \log \left( 1 + \frac{\exp \sqrt{s}}{n!} \right) = O(N^2) + N \sqrt{s} - \sum_{n=1}^{N} n \log n.
\]
The function $x \log x$ is a monotonically increasing positive function when $x > 1$, so comparing our sum with the area under a graph shows that
\[
\int_1^{N+1} x \log x \, dx < \sum_{n=1}^{N} n \log n < \int_2^{N+1} x \log x \, dx.
\]
Since $\int x \log x \, dx = \frac{1}{2} x^2 \log x - \frac{1}{4} x^2$, it follows that
\[
\sum_{n=1}^{N} n \log n = \frac{1}{2} N^2 \log N + O(N^2),
\]
and therefore
\[
\sum_{n=1}^{\infty} \log \left(1 + \frac{e^{\sqrt{s}}}{n!}\right) = O(N^2) + N \sqrt{s} - \frac{1}{2} N^2 \log N.
\]
Finally, we need to make the relationship between $N$ and $s$ explicit. The definition of $N$ says that
\[
\log N! \leq \sqrt{s} < \log(N + 1)!,
\]
so (again by Stirling’s formula)
\[
\sqrt{s} \sim N \log N, \quad \text{and} \quad N \sqrt{s} \sim N^2 \log N.
\]
Combining this information with our previous growth estimate shows that
\[
\sum_{n=1}^{\infty} \log \left(1 + \frac{e^{\sqrt{s}}}{n!}\right) = \left(\frac{1}{2} N^2 \log N\right) (1 + o(1)).
\]
To rewrite the right-hand side in terms of $s$, observe that
\[
\frac{1}{2} N^2 \log N = \frac{(N \log N)^2}{2 \log N} \sim \frac{s}{2 \log N}.
\]
Moreover, since $\sqrt{s} \sim N \log N$, it follows that
\[
\frac{1}{2} \log s - \log N - \log \log N \to 0, \quad \text{so} \quad \log s \sim 2 \log N.
\]
Thus
\[
\sum_{n=1}^{\infty} \log \left(1 + \frac{e^{\sqrt{s}}}{n!}\right) = \frac{s}{\log s} (1 + o(1)),
\]
as claimed. \qed