Exercises for Section 3.1

9 a) Show $\beta 0 = 0$ for each scalar $\beta$. We must show $\beta 0 + x = x$ for all vectors $x$. For then, $\beta 0$ must be 0 since the Zero in the vector space is unique.

Now if $\beta = 0$, then we are done since $0x = 0$ for all vectors $x$ (including $x = 0$) by Theorem 3.1.1.

If $\beta \neq 0$, then

$$\beta 0 + x = \beta 0 + \beta \left( \frac{1}{\beta} x \right)$$

$$= \beta (0 + \frac{1}{\beta} x)$$

$$= \beta \left( \frac{1}{\beta} x \right) \text{ by definition of 0}$$

$$= x$$

Therefore, $\beta 0 + x = x$, as desired.

b) We are to show that if $\alpha x = 0$, and if $\alpha \neq 0$, then $x = 0$. We have

$$x = \frac{1}{\alpha} \alpha x$$

$$= \frac{1}{\alpha} 0$$

$$= 0 \text{ by part a)}$$

10 Axiom 6 does not hold because

$$(\alpha + \beta)(x_1, x_2) = ((\alpha + \beta)x_1, (\alpha + \beta)x_2)$$
On the other hand
\[ α(x_1, x_2) ⊕ β(x_1, x_2) = (αx_1, αx_2) ⊕ (βx_1, βx_2) = (αx_1 + βx_1, 0) \]

13 This space is NOT a vector space. First, there is no 0 vector for this space since there is no smallest real number. In addition, axiom 6 does not hold since \((1 + -1)x = 0x = 0\). However \(x ⊕ (-x) = x\) if \(x > 0\).

Section 3.2

18 Suppose \(U\) and \(V\) are subspaces of the vector space \(W\). We are to show that \(U \cap V\) is a subspace. Therefore, we must show

1) If \(u \in U \cap V\) and \(r \in R\), then \(ru \in U \cap V\).

2) If \(u\), and \(v \in U\) then \(u + v \in U \cap V\).

For the first statement, if \(u \in U \cap V\), then \(u\) belongs to both subspaces \(U\) and \(V\). Therefore \(ru\) belongs to both \(U\) and \(V\); thus \(ru\) lies in \(U \cap V\).

For the second statement if \(u, v \in U \cap V\), then \(u\) and \(v\) belong to both subspaces \(U\) and \(V\). Therefore \(u + v\) belongs to both \(U\) and \(V\); thus \(u + v\) lies in \(U \cap V\).

Therefore statements 1) and 2) hold and so \(U \cap V\) is a subspace.

19 \(S \cup T\) consist of the union of the \(x\) and \(y\) coordinates axes. This is NOT a subspace since it is not closed under addition. For example, both \((1, 0)\) and \((0, 1)\) belong to \(S \cup T\), but \((1, 0) + (0, 1) = (1, 1)\) which does NOT belong to \(S \cup T\).