Solutions to Chapter 3

1. Summing the two trig identities that are given in the problem yields

\[
\cos mt \cos \lambda t = \frac{(\cos(\lambda + m)t + \cos(\lambda - m)t)}{2}
\]

Integrating the expression on the right gives

\[
\int_{-\pi}^{\pi} \cos mt \cos \lambda t \, dt = \frac{1}{2} \left( \frac{\sin(\lambda + m)t}{\lambda + m} + \frac{\sin(\lambda - m)t}{\lambda - m} \right) \bigg|_{-\pi}^{\pi}
\]

Inserting the limits and simplifying yields

\[
\frac{-2((-1)^m \lambda \sin(\lambda \pi))}{(m^2 - \lambda^2)}
\]

Setting \( m = 3 \) shows that the Fourier transform of \( \cos(3t) \) (from \( -\pi \) to \( \pi \)) is

\[
\frac{\sqrt{2} \lambda \sin(\pi \lambda)}{\sqrt{\pi}(-9 + \lambda^2)}
\]

as claimed.

2. Since \( \sin \) is an odd function, we only need to compute the sine part of the Fourier transform. That is, we must compute

\[
\left. \frac{-i}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sin 3t \sin \lambda t \, dt \right|
\]

Subtracting the two trig identities given in problem 1, gives

\[
\frac{-i}{\sqrt{2\pi}} \sin 3t \sin \lambda t = \frac{-i}{\sqrt{2\pi}} (\cos(\lambda - 3)t - \cos(\lambda + 3)t) / 2
\]

which when integrated from \( -\pi \) to \( \pi \) gives

\[
\frac{i}{2\sqrt{2\pi}} \left( \frac{\sin(\lambda - 3)t}{\lambda - 3} - \frac{\sin(\lambda + 3)t}{\lambda + 3} \right) \bigg|_{-\pi}^{\pi} = \frac{i\sqrt{2}\sin(\pi \lambda)}{\sqrt{\pi}(-3 + \lambda)(3 + \lambda)}
\]

3. The Maple code for displaying \( f \) and computing its Fourier transform is given below.

```maple
> f:=t-> piecewise(t<-2,0,t<-1,t^2+4*t+4,t<1,2-t^2,t<2,t^2-4*t+4,0);
f := t -> piecewise(t < -2, 0, t < -1, t^2+4*t+4, t < 1, 2-t^2, t < 2, t^2-4*t+4, 0)
```
Note that $f$ is continuous. Each piece matches up at the boundary points. For example the limit of $t^2 + 4t + 4$ as $t$ approaches $-2$ is zero, which is the same as the value of $f$ on the interval $t < -2$. The other pieces match up similarly.

The derivative is given by

$$D(f)(t); \begin{cases} 0 & t < -2 \\ t^2 + 4t + 4 & t < -1 \\ 2 - t^2 & t < 1 \\ t^2 - 4t + 4 & t < 2 \\ 0 & \text{otherwise} \end{cases}$$

The derivative is also continuous since each of its pieces match up at the boundary defining points. For example, the limit of $2t + 4$ as $t$ approaches $-2$ is 0 which agrees with $f'$ for $t < -2$.

Here is the computation of the Fourier transform. Since $f$ is even, we need only compute the cosine part of the transform.

$$\frac{1}{\sqrt{2\pi}} \int f(t) \cos(\lambda t) \, dt; \text{value}(\text{"});$$
\[ \sqrt{2} \int_{-2}^{2} \left( \begin{array}{cc} 0 & t < -2 \\ t^2 + 4t + 4 & t < -1 \\ 2 - t^2 & t < 1 \\ t^2 - 4t + 4 & t < 2 \\ 0 & \text{otherwise} \end{array} \right) \cos(\lambda t) \, dt \]

\[ \frac{1}{\sqrt{\pi}} \]

\[ \cos(\lambda) (\cos(\lambda) - 1) \]

\[ \frac{1}{\sqrt{\pi}} \lambda^3 \]

\[ \text{trans} := -4 \frac{\sqrt{2} \sin(\lambda) (\cos(\lambda) - 1)}{\sqrt{\pi} \lambda^3} \]

Here is the plot of the transform over the interval \(-10 \leq \lambda \leq 10\).

\[ \text{plot}(\text{trans}, \lambda = -10..10); \]

Note that the Fourier transform decays like \(1/\lambda^3\). There is a pattern here. A function which has finite support and bounded leads to a Fourier transform that decays like \(1/\lambda\) (see example 21 in the text, which was also done in class). If the function is also continuous then its transform decays like \(1/\lambda^2\) (such as in problem 1). If the function also has a continuous derivative, then the transform decays like \(1/\lambda^3\) as in this exercise.

5. To prove the \(\mathcal{F}^{-1}\) is linear, we have

\[ \mathcal{F}^{-1}[f + g](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(\lambda) + g(\lambda)]e^{i\lambda x} \, d\lambda \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(\lambda)]e^{i\lambda x} \, d\lambda + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(\lambda)]e^{i\lambda x} \, d\lambda \]
\[ = \mathcal{F}^{-1}[f](x) + \mathcal{F}^{-1}[g](x) \]
To prove
\[ F^{-1}[\lambda^n f(\lambda)](t) = (-i)^n \frac{d^n}{dt^n} \{ t^{-1} f(t) \} \]

We start with the right side:
\[
(-i)^n \frac{d^n}{dt^n} \{ F^{-1}[f(t)] \} = (-i)^n \frac{d^n}{dt^n} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(\lambda)] e^{i\lambda t} d\lambda \right\}
\]
\[
= (-i)^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(\lambda)] \frac{d^n}{dt^n} \{ e^{i\lambda t} \} d\lambda
\]
\[
= (-i)^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(\lambda)] (i\lambda)^n e^{i\lambda t} d\lambda
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(\lambda)] (i\lambda)^n e^{i\lambda t} d\lambda
\]
\[
= F^{-1}[\lambda^n f(\lambda)](t)
\]

To prove
\[ F^{-1}[f^{(n)}(\lambda)](t) = (-it)^n F^{-1}[f](t) \]
we start with the left side with \( n = 1 \):
\[
F^{-1}[f'(\lambda)](t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(\lambda) e^{i\lambda t} d\lambda
\]

We integrate by parts with \( dv = f' \) and \( u = e^{i\lambda t} \) to obtain
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(\lambda) e^{i\lambda t} d\lambda = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\lambda) e^{i\lambda t} d\lambda
\]
\[
- \frac{1}{\sqrt{2\pi}} \left( f(\lambda) e^{i\lambda t} \right)_{\lambda = -\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\lambda) (ite^{i\lambda t}) d\lambda
\]

The boundary terms are zero since \( f(\lambda) = 0 \) for large \( \lambda \). Therefore
\[
F^{-1}[f'(\lambda)](t) = - \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} f(\lambda) (ite^{i\lambda t}) d\lambda = -it F^{-1}[f(\lambda)](t)
\]

To show the statement regarding the Laplace transform, we have
\[
\mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) e^{-i\lambda t} dt
\]

Since \( f(t) = 0 \) for \( t < 0 \), the lower limit of integration starts at 0 rather than \(-\infty\). On the other hand
\[
\frac{1}{\sqrt{2\pi}} \mathcal{L} f(i\lambda) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) e^{-it\lambda} dt
\]
which is the same as \( \mathcal{F}[f](\lambda) \).

8. Here is the proof of Theorem 3.12
Shifts. We have

\[ \hat{z}_j = \sum_{k=0}^{n-1} z_k \omega^{kj} \quad \text{where } w = e^{2\pi i/n} \]

\[ = \sum_{k=0}^{n-1} y_{k+1} \omega^{kj} \]

We change indices by letting \( \ell = k + 1 \) to obtain

\[ \hat{z}_j = \sum_{\ell=1}^{n} y_{\ell} \omega^{(\ell-1)j} \]

\[ = w^{-j} \sum_{\ell=1}^{n} y_{\ell} \omega^{j\ell} \]

\[ = w^{-j} \sum_{\ell=1}^{n} y_{\ell} \omega^{j\ell} \]

where the last equality holds since \( w^{-1} = w \) (recall that \( w = e^{2\pi i/n} \)). The last line is \( w^j \hat{y}_j \) as desired.

Convolutions. We must show

\[ (x \ast y)_{k+n} = (x \ast y)_k \]

The left side is

\[ (x \ast y)_{k+n} = \sum_{j=0}^{n-1} x_j y_{k+n-j} \]

Since \( y_{k+n-j} = y_{k-j} \), we have

\[ (x \ast y)_{k+n} = \sum_{j=0}^{n-1} x_j y_{k+n-j} = \sum_{j=0}^{n-1} x_j y_{k-j} = (x \ast y)_k \]

Convolution Theorem. We have

\[ (x \ast y)_\ell = \sum_{k=0}^{n-1} (x \ast y)_k \omega^{\ell k} \]

\[ = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} x_{k-j} y_j \omega^{\ell k} \]

\[ = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} x_{k-j} \omega^{\ell(k-j)} y_j \omega^{j} \]

\[ = \sum_{j=0}^{n-1} \sum_{t=-j}^{n-1} x_{t} \omega^{\ell t} y_j \omega^{j} \] with \( t = k - j \)

\[ = \sum_{j=0}^{n-1} \sum_{t=0}^{n-1} x_{t} \omega^{\ell t} y_j \omega^{j} \]

\[ = \sum_{j=0}^{n-1} \sum_{t=0}^{n-1} x_{t} \omega^{\ell t} y_j \omega^{j} \]

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where the last equation follows from the fact that $x_{q-n} = x_q$ and $w^{(q-n)\ell} = w^{q\ell}$ (so the inner sum from $t = -j$ to $t = -1$ is the same as the sum from $t = n - j$ to $n - 1$). The last expression on the right is

$$\sum_{t=0}^{n-1} x_t w^{\ell t} \sum_{j=0}^{n-1} y_j w^{j\ell} = \tilde{x}_\ell \tilde{y}_\ell$$

as desired.

Last Property. We have

$$\hat{y}_{n-k} = \sum_{j=0}^{n-1} y_j w^{j(n-k)}$$

Now use the fact that $w^n = 1$ and that $w^{-jk} = w^{jk}$ (again, recall that $w = e^{2\pi i / n}$). We obtain

$$\hat{y}_{n-k} = \sum_{j=0}^{n-1} y_j w^{jk}$$

Since the $y_j$ are real, the right side is

$$\sum_{j=0}^{n-1} y_j w^{jk} = \hat{y}_k$$

as desired.