Exam III: Night Before Drill

1. How long will it take for an investment earning 5.4% interest, compounded continuously, to triple?

\[ A = Pe^{rt} \]

\[ 3P = Pe^{0.054t} \]

\[ \ln 3 = \ln e^{0.054t} \]

\[ \frac{\ln 3}{0.054} = t \]

\[ t \approx 20.345 \text{ yrs} \]

2. Find \( f[g(x)] \) and \( g[f(x)] \) given

(a) \( f(x) = e^{x+2}; \quad g(x) = \sqrt{x+7} \)

\[ f[g(x)] = f(\sqrt{x+7}) = e^{\sqrt{x+7} + 2} \]

\[ g[f(x)] = g(e^{x+2}) = \sqrt{e^{x+2} + 7} \]

(b) \( f(x) = x^2 + 8; \quad g(x) = \ln x \)

\[ f[g(x)] = f(\ln x) = (\ln x)^2 + 8 \]

\[ g[f(x)] = g(x^2 + 8) = \ln(x^2 + 8) \]
3. Find the interval(s) where \( f(x) = e^{x^3+2x^2-4x} \) is increasing/decreasing.

**Domain of \( f \): \( \mathbb{R} \)**

\[
f'(x) = e^{x^3+2x^2-4x} \left( 3x^2+4x-4 \right) = 0
\]

\[
e^{x^3+2x^2-4x} \neq 0 \quad 3x^2+4x-4 = 0
\]

\[
(3x-2)(x+2) = 0
\]

\[
x = \frac{2}{3} \quad x = -2 \quad \leftarrow CV's
\]

**INCREASING:** \(-\infty, -2\) \( \cup \) \( \frac{2}{3}, \infty \)

**DECREASING:** \(-2, \frac{2}{3}\)
4. Find intervals, if any exist, where the following functions are concave up/concave down and determine any points of inflection.

(a) \( f(x) = \ln\left(\frac{x^2}{2}\right) \quad D: x^2 > 0 \Rightarrow x \neq 0 \)

\[
f'(x) = \left(\frac{1}{x^2}\right) \cdot (2x) = \frac{2}{x} = 2x^{-1}
\]

\[
f''(x) = -2x^{-2} = \frac{-2}{x^2} = 0
\]

\[
f'' \text{ undefined at } x = 0 \quad \text{(not in domain)}
\]

(b) \( f(x) = (5x)e^x \quad D: \mathbb{R} \)

\[
f'(x) = 5e^x + 5xe^x = (5e^x)(1 + x)
\]

\[
f''(x) = 5e^x(1 + x) + 5e^x(1) = 5e^x[(1 + x) + 1] = 5e^x(x + 2)
\]

\[
\begin{array}{c|cc}
\text{f''} & - & 0 & + \\
\hline
\text{f} & - & 0 & + \\
\end{array}
\]

CV: \((-\infty, 0) \quad (0, \infty)\)

CP: never

NO Inf. Pts.

CV: \((-\infty, -2) \quad [-2, \infty) \quad (-2, f(-2))\)
5. The cost of producing $x$ units of a product is given by $C(x) = 800 + 200x - 100 \ln x$ for $x \geq 1$. Find the minimum average cost.

$$\bar{C}(e^9) \approx 1620.516.79$$

$$\text{Avg. Cost} = \bar{C} = \frac{C}{x} = \frac{800 + 200x - 100 \ln x}{x}$$

$$\bar{C}' = \left(\frac{x}{x^2} \right) \left[ 200 - 100 \left(\frac{1}{x}\right) \right] - \left[ 800 + 200x - 100 \ln x \right] \left(\frac{1}{x^2} \right)$$

$$= \frac{200x - 100 - 800 - 200x + 100 \ln x}{x^2} = \frac{-900 + 100 \ln x}{x^2}$$

$$\bar{C}' = 0 = -\frac{900 + 100 \ln x}{x^2}$$

$$\Rightarrow -900 + 100 \ln x = 0$$

$$100 \ln x = 900$$

$$\ln x = 9 \Rightarrow x = e^9$$

$$\Rightarrow \text{Only one critical value in interval, so will either be abs max or abs min.}$$

$$\bar{C}'' = \left(\frac{x^4}{x^4} \right) \left(100 \left(\frac{1}{x}\right) - (-900 + 100 \ln x) \left(\frac{1}{x^2} \right) \right)$$

$$= \frac{100x + 1800x - 200x \ln x}{x^4}$$

$$= \frac{1900x - 200x \ln x}{x^4}$$

$$\bar{C}''(e^9) = + \Rightarrow \bar{C}(e^9) = +$$
6. Find \(\frac{dy}{dx}\) given the following:

(a) \(y = x^2 e^{x^2 + 4x} + 3^2\)

\[
\frac{dy}{dx} = (2x)e^{x^2 + 4x} + x^2 \left[ e^{x^2 + 4x} (2x + 4)(\ln 3) \right] + 0
\]

(b) \(y = \log_{10} \left( \ln \left( x^2 + 9 \right) \right) + \ln 5\)

\[
\frac{dy}{dx} = \left( \frac{1}{\ln(x^2 + 9)} \right) \left( \frac{2x}{x^2 + 9} \right) \left( 2x \right) \left( \frac{1}{\ln 10} \right) + 0
\]

(c) \(y = e^u; \ u = -2x^2\)

\[
\text{Method 1}
\]

\[
\frac{dy}{du} = e^u, \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

\[
\frac{du}{dx} = -4x, \quad \frac{dy}{dx} = e^u (-4x) = e^{-2x^2} (-4x)
\]

\[
\text{Method 2}
\]

\[
\frac{dy}{dx} = e^{-2x^2} (-4x)
\]
7. Given the price-demand equation \( x = 10(p - 9)^2 \),

(a) Find the elasticity of demand, \( E(p) \).

\[
E(p) = -p \frac{f'(p)}{f(p)} = -p \left\{ \frac{\frac{2}{5}(p - 9)}{10(p - 9)^2} \right\} = \frac{-2p}{p - 9}
\]

(b) If \( p = 5 \), is demand elastic, inelastic, or is there unit elasticity?

\[
E(5) = \frac{-2(5)}{5 - 9} = \frac{-10}{-4} = \frac{10}{4} > 1 \implies \text{Elastic}
\]

(c) Should the price be raised, lowered, or kept at \( 5 \) in order to increase revenue?

\( \text{Elastic} \implies \text{lower the price} \)

(d) At what price will revenue be maximized?

\[
E(p) = 1 \implies \frac{-2p}{p - 9} = 1 \implies -2p = p - 9 \implies -3p = -9 \implies p = 3
\]
8. Evaluate the following:

(a) \( \int \left( 2x + \frac{5}{x^3} - x^{5/4} - e \right) dx \)

\[ \int \left( 2x^4 + 5x^{-3} - x^{5/4} - e \right) dx \]

\[ = 2 \left( \frac{x^2}{2} \right) - 5 \left( \frac{x^{-2}}{-2} \right) - \frac{9}{4} x^{9/4} - e x + C \]

\[ = \frac{x^2}{2} - \frac{5}{2} x^{-2} - \frac{9}{4} x^{9/4} - e x + C \]

(b) \( \int \frac{y^2 - \sqrt{y}}{y^3} dy = \int \left( \frac{y^2}{y^3} - \frac{y^{1/2}}{y^3} \right) dy = \int \left( \frac{1}{y} - y^{-5/2} \right) dy \)

\[ = \ln |y| - \frac{2}{3} y^{-3/2} + C \]

\[ = \ln |y| + \frac{2}{3} y^{-3/2} + C \]
(c) \[ \int u^2 \, dt \]  
\[ u = 4t \]  
\[ du = 4 \, dt \]  
\[ \frac{1}{4} \, du = dt \]  
\[ \int e^{u/4} \, du = \frac{1}{4} e^u + C \]  
\[ \frac{1}{4} e^u + C \] 

(d) \[ \int \frac{3}{5x \ln 4x} \, dx \]  
\[ u = \ln \frac{4x}{x} \]  
\[ \frac{3}{5} \, (du) = (\frac{1}{4x}) \, dx \]  
\[ \frac{3}{5} \, du = \frac{3}{5x} \, dx \]  
\[ \int \frac{3}{5} \, du = \frac{3}{5} \ln |u| + C \]  
\[ \frac{3}{5} \ln |\ln 4x| + C \]
(e) \[ \int (10x - 20)e^{2x-4x} \, dx \]
\[ u = x^2 - 4x \]
\[ du = (2x-4) \, dx \]
\[ 5du = (10x-20) \, dx \]
\[ \int 5e^u \, du = 5e^{x^2 - 4x} + C \]

(f) \[ \int t\sqrt{2t+5} \, dt = \int \sqrt{t(2t+5)} \, dt \]
\[ u = 2t + 5 \]
\[ du = 2 \, dt \]
\[ dt = \frac{1}{2} \, du \]
\[ \int \frac{(u-5)^{\frac{1}{2}}}{2} \, du \]
\[ \int \frac{1}{4} (u-5)^{\frac{1}{2}} \, du = \frac{1}{4} \int (u^{3/2} - 5u^{1/2}) \, du = \frac{1}{4} \left[ \frac{u^{5/2}}{5/2} - 5 \frac{u^{3/2}}{3/2} \right] + C \]
\[ \frac{1}{4} \left[ \frac{2}{5} (2t+5)^{5/2} - 10 \frac{(2t+5)^{3/2}}{3} \right] + C \]
\[ \int \frac{1}{u^3} \, du = \frac{1}{2} \frac{-1}{u} + C \]

9. Find the cost function for a tape manufacturer, if the marginal cost, in dollars/case, is given by \(150 - 0.1e^x\), where \(x\) is the number of cases of tape produced and the manufacturer has $100 worth of fixed costs.

\[ C = \int C' \, dx = \int (150 - 0.1e^x) \, dx = 150x - 0.1e^x + K \]

\[ C(0) = 150(0) - 0.1e^0 + K = 100 \]

\[ -0.1 + K = 100 \rightarrow K = 100.1 \]

\[ \Rightarrow C(x) = 150x - 0.1e^x + 100.1 \]
10. Given \( \int_{2}^{5} (-x^2 + 4) \, dx \)

(a) Sketch the region indicated by this integral.

\[-x^2 + 4 = 0 \quad \therefore x = \pm 2\]

(b) Approximate the value of the integral by finding the left hand and right hand Riemann sums with 4 rectangles and then again with 100 rectangles. Draw pictures of the areas being found with 4 rectangles.

\[
\text{Width of rect } \Delta x = \frac{b-a}{n} = \frac{5-2}{4} = \frac{3}{4}
\]

\[
\begin{array}{c|c|c|c|c|c|c}
\hline
 & x & 2 & 2.75 & 3.5 & 4.25 & 5 \\
\hline
x & -x^2 + 4 & 0 & -3.5625 & -8.25 & -14.0625 & -21 \\
\hline
\end{array}
\]

\[
\text{LHS } = \frac{3}{4} \left( 0 - 3.5625 - 8.25 - 14.0625 \right) = -19.40625
\]

\[
\text{RHS } = \frac{3}{4} \left( -3.5625 - 8.25 - 14.0625 - 21 \right) = -35.15625
\]

(c) Find the exact value of the integral.

\[
\int_{2}^{5} (-x^2 + 4) \, dx = \left[ -\frac{x^3}{3} + 4x \right]_{2}^{5} = \left[ -27 \right]
\]

(d) How much area is there between the curve \( y = -x^2 + 4 \) and the x-axis from \( x = 2 \) to \( x = 5 \)?

\[27 \text{ Sq. units}\]
11. Evaluate the following EXACTLY:

(a) \( \int_a^b \left( x^3 + \frac{1}{x} \right) \, dx \quad (0 < a < b) \)

\[
\begin{align*}
= \frac{x^4}{4} + \ln |x| \bigg|_a^b \\
= \left( \frac{b^4}{4} + \ln |b| \right) - \left( \frac{a^4}{4} + \ln |a| \right) \\
= \frac{b^4}{4} - \frac{a^4}{4} + \ln |b| - \ln |a| = \frac{b^4}{4} - \frac{a^4}{4} + \ln \left| \frac{b}{a} \right| 
\end{align*}
\]

(b) \( \int_2^4 (e^t + \pi) \, dt \)

\[
\begin{align*}
&= \left. e^t + \pi t \right|_2^4 \\
&= (e^4 + 4\pi) - (e^2 + 2\pi) = \frac{e^4 - e^2 + 2\pi}{2} 
\end{align*}
\]
(e) \[ \int_{1}^{3} \frac{x^2 + 1}{x^3 + 3x} \, dx \]

\[ u = x^3 + 3x \quad \Rightarrow \quad x = 1 \rightarrow u = 1^3 + 3(1) = 4 \]

\[ x = 3 \rightarrow u = 3^3 + 3(3) = 36 \]

\[ du = (3x^2 + 3) \, dx \]

\[ du = 3(x^2 + 1) \, dx \]

\[ \frac{1}{3} \, du = (x^2 + 1) \, dx \]

\[ \int_{4}^{36} \frac{1}{3} \left( \frac{1}{u} \right) \, du = \frac{1}{3} \ln |u| \bigg|_{4}^{36} = \frac{1}{3} \ln |36| - \frac{1}{3} \ln |4| \]

\[ = \frac{1}{3} \left[ \ln 36 - \ln 4 \right] \]

\[ = \frac{1}{3} \ln \left( \frac{36}{4} \right) = \frac{1}{3} \ln 9 \]

12. Write a definite integral to indicate the shaded area in the graph below.

\[ \text{Area} = \int_{-3}^{4} f(x) \, dx \]

\[ = \int_{3}^{2} 6 \, dx + \int_{2}^{4} (2x + 2) \, dx \]

\[ = \frac{10 - 6}{4 - 2} = \frac{4}{2} = 2 \quad y - 6 = 2(x - 2) \quad \Rightarrow \quad y = 2x + 2 \]
13. Suppose copper is being extracted from a mine at a rate given by \( y = 100e^{-0.2t} \), where \( t \) is the number of years since mining began and \( y \) is measured in tons of copper/year. At this rate, how much copper will be extracted during the third year of mining?

\[
\int_{2}^{3} 100e^{-0.2t} \, dt \rightarrow \frac{1}{0.2} \ln \left( \frac{100e^{-0.2(3)}}{100e^{-0.2(2)}} \right) \approx 60.75 \text{ tons.}
\]

14. If the temperature \( C(t) \) in an aquarium is made to change according to \( C(t) = t^3 - 2t + 10 \) for \( 0 \leq t \leq 2 \) (in degrees Celsius), what is the average temperature over the period of time for which the temperature is regulated?

\[
\frac{1}{2-0} \int_{0}^{2} (t^3 - 2t + 10) \, dt = \frac{1}{2} \ln \left( \frac{t^3 - 2t + 10}{0} \right) \right) \approx 10^\circ C
\]
15. Find the area between \( y = x^2 + 1 \) and \( y = -x^2 + 19 \) on \([0, 5]\).

Intersection Point(s)

\[
y_1 = y_2 \\
x^2 + 1 = -x^2 + 19 \\
2x^2 = 18 \\
x^2 = 9 \rightarrow x = \pm 3
\]

\[
\text{Area} = \int_{0}^{3} (y_2 - y_1) \, dx + \int_{3}^{5} (y_1 - y_2) \, dx
\]

\[
= \text{fnInt} (y_2 - y_1, x, 0, 3) + \text{fnInt} (y_1 - y_2, x, 3, 5)
\]

\[
= \frac{196}{3}
\]
16. Find the area bounded by $y = x^3$ and $y = x$.

Intersection Point(s)

$x^2 = x$
$x^3 - x = 0$
$x(x^2 - 1) = 0$
$x = 0, x = \pm 1$

$$\text{Area} = \int_{-1}^{0} (y_1 - y_2) \, dx + \int_{0}^{1} (y_2 - y_1) \, dx$$

$$= \text{fnInt} (y_1, y_2, x, -1, 0) + \text{fnInt} (y_2 - y_1, x, 0, 1)$$

$$= \frac{1}{2}$$
17. If supply and demand for a product are given by \( p = 5 + 0.004x^2 \) and \( p = 25 - 0.004x^2 \), respectively, find the following at equilibrium price:

(a) consumers’ surplus

(b) producers’ surplus

\[
\begin{align*}
\text{Supply} & \quad \text{Demand} \\
\text{CS} & = \int_0^{50} \left[ (25 - 0.004x^2) - 15 \right] \, dx \\
& = \left. \left[ 10x - \frac{1}{2} \cdot \frac{0.004}{3}x^3 \right] \right|_0^{50} \\
& = 333.33 \\
\text{PS} & = \int_0^{50} \left[ 15 - (5 + 0.004x^2) \right] \, dx \\
& = \left. \left[ 10x - \frac{1}{2} \cdot \frac{0.004}{3}x^3 \right] \right|_0^{50} \\
& = 333.33
\end{align*}
\]

\[ p = 5 + 0.004(50)^2 = 15 \]

E0 PT: \( S = D \)

\[
\begin{align*}
5 + 0.004x^2 & = 25 - 0.004x^2 \\
0.008x^2 & = 20 \\
x^2 & = 2500 \\
x & = \pm 50
\end{align*}
\]
18. The probability that a particular doctor will spend $t$ hours with a patient during an office visit is given by the probability density function $f(t) = \begin{cases} \frac{4}{3}(t+1)^{-2}, & 0 \leq t \leq 3 \\ 0, & \text{otherwise} \end{cases}$

(a) What is the probability that this doctor will spend more than 1 hour with a randomly selected patient?

(b) What is the probability that this doctor will spend exactly 1 hour with a randomly selected patient?

\[ a) \int_{1}^{3} \frac{4}{3}(t+1)^{-2} \, dt = \frac{4}{3} \ln(\frac{4}{3}) - (t+1)^{-1} \bigg|_{1}^{3} = 0.3333 \]

\[ b) \int_{1}^{1} \frac{4}{3}(t+1)^{-2} \, dt = 0 \]
19. Starting at age 25, you deposit $1500 a year into a retirement account. If the deposits are treated as a continuous income stream and the money in the account earns 6% compounded continuously,

(a) How much will you have in the account if you retire at age 65?

(b) How much of the final amount is interest?

\[ FV = \int_{0}^{40} 1500 e^{-0.06t} \, dt \]

\[ \approx \$250,579.41 \]

(b) \[ \text{Deposited} = 1500 \times 40 \]

\[ \text{Interest} = \frac{250,579.41}{1500 \times 40} \]

\[ \approx \$190,579.41 \]