2.1 - Polynomial and Rational Functions

Polynomial Functions

**Def:** A polynomial function of *degree* \( n \) has the form

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \]

where \( a_0, a_1, \ldots, a_n \) are real numbers with \( a_n \neq 0 \) and \( n \) is a non-negative integer. The coefficient \( a_n \) is known as the **leading coefficient** of the polynomial.

\[ n = 0 : f(x) = a_0 \]
\[ n = 1 : f(x) = a_1 x + a_0 \]
\[ n = 2 : f(x) = a_2 x^2 + a_1 x + a_0 \]
\[ n = 3 : f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \]
\[ n = 4 : f(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \]

**Ex:** Which of the following are polynomials? For each polynomial, give its degree and circle the leading coefficient.

(a) \( f(x) = 2 - x^2 - 5x^8 \)

(b) \( g(x) = x^{1/4} + x + 4 \)

(c) \( h(x) = 3x + x^4 - \pi \)

(d) \( k(x) = 2x^2 - \sqrt{x} \)

(e) \( n(x) = 5x^3 - x^2 + x^{-1} \)

**Odd-Degree Polynomials**

**Even-Degree Polynomials**
Properties of Polynomials

- Domain is \((-\infty, \infty)\) for all polynomials.
- Polynomials are continuous - they have no holes or breaks in their graphs.
- The graphs of polynomials have no sharp corners.
- The graph of a polynomial function of positive degree \(n\) can have at most \(n - 1\) turning points (a point separating an increasing portion from a decreasing portion) and can cross the \(x\)-axis at most \(n\) times.

Ex: For each graph below, answer the following:

(a) How many turning points are on the graph?

(b) What is the minimum degree of a polynomial function that could have the graph?

(c) Is the leading coefficient of the polynomial positive or negative?

Regression Polynomials

In order to find the best fitting polynomial model to a set of data, use the following options under STAT on your calculator:

- 4:LinReg\((ax+b)\) for a linear model \(y = ax + b\)
- 5:QuadReg for a quadratic model \(y = ax^2 + bx + c\)
- 6:CubicReg for a cubic model \(y = ax^3 + bx^2 + cx + d\)
- 7:QuartReg for a quartic model \(y = ax^4 + bx^3 + cx^2 + dx + e\)
Rational Functions

Def: A **rational function** has the form

\[ f(x) = \frac{n(x)}{d(x)} \]

where \( n(x) \) and \( d(x) \) are polynomials and \( d(x) \neq 0 \). The domain of a rational function is the set of all real numbers such that \( d(x) \neq 0 \).

**Asymptotes**

- **Finding Vertical Asymptotes:**

For the rational function \( f(x) = \frac{n(x)}{d(x)} \), if there is a value \( a \) that makes the denominator zero \((d(a) = 0)\) and the numerator NOT zero \((n(a) \neq 0)\), then \( x = a \) is a **vertical asymptote** (VA).

**NOTE:** If there is a value \( a \) which makes BOTH the numerator and denominator of the rational function \( f(x) = \frac{n(x)}{d(x)} \) zero, then there is a “hole” in the graph of \( f(x) \) when \( x = a \).

**Ex:** Identify any holes or VA which occur in the graphs of the following functions:

(a) \( f(x) = \frac{x^2 - x - 2}{x^2 - 9} \)

(b) \( g(x) = \frac{2x}{x - 4} \)

(c) \( h(x) = \frac{x^2 + 6x + 8}{x^2 - x - 6} \)
• Finding Horizontal Asymptotes:

** Horizontal asymptotes (HA) describe the end behavior of a function **

For the rational function \( f(x) = \frac{n(x)}{d(x)} \):

– If the degree of \( n(x) \) is greater than the degree of \( d(x) \), then there is NO HA.
– If the degree of \( d(x) \) is greater than the degree of \( n(x) \), then there is a HA of \( y = 0 \).
– If the degrees of \( n(x) \) and \( d(x) \) are equal, then there is a HA of \( y = \frac{a}{b} \), where \( a \) and \( b \) are the leading coefficients of \( n(x) \) and \( d(x) \), respectively.

**Ex:** Find the HA of the following functions:

(a) \( m(x) = \frac{4x^3 + x^2 + 1}{2x^2 + 5x^4} \)

(b) \( n(x) = \frac{-2x^2 + 4}{x + 5} \)

(c) \( r(x) = \frac{3x^8 + 4x^6 - 3x^3 + 1}{2 + 5x^2 - 7x^8} \)

**Ex:** Determine the \( x \) and \( y \)-intercepts and all asymptotes of the following function and then make a sketch of the function.

\[
f(x) = \frac{(2x + 1)(x - 4)}{(x + 3)(x - 5)}
\]