4.2 - Second Derivatives and Graphs

Any derivative beyond the first derivative is called a higher-order derivative.

**Notation:** First Derivative  Second Derivative  $n^{th}$ Derivative

\[
\begin{array}{ccc}
\frac{dy}{dx} & \frac{d^2y}{dx^2} & \frac{d(n)y}{dx^n} \\
\end{array}
\]

**Ex:** Find the first three derivatives of the following functions.

(a) $g(t) = 2t^{10} - t^6 + 5$

(b) $f(x) = 5\sqrt{x} + 2x$

**Ex:** Given $f(x) = \frac{5}{3}x^3 + 2.5x^2 + 4x + 6$, determine the intervals where $f'(x)$ is increasing and decreasing.
Test for Concavity: For a function whose second derivative exists on an open interval \((a, b)\):

1. If \(f''(x) > 0\) (\(f'\) is increasing) for all \(x\) on \((a, b)\), then \(f(x)\) is concave up \((\sim)\) on \((a, b)\).

2. If \(f''(x) < 0\) (\(f'\) is decreasing) for all \(x\) on \((a, b)\), then \(f(x)\) is concave down \((\sim)\) on \((a, b)\).

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**Def:** An inflection point is a point \((d, f(d))\) where the graph of \(f(x)\) changes concavity.

Locating Inflection Points:

1. Determine the values of \(x\) where the second derivative is zero or where the second derivative is undefined.

2. Place these values on a number line and make a sign chart for the second derivative.

3. The point is an inflection point if the second derivative changes sign and if the \(x\)-value is in the domain of \(f(x)\).

**Ex:** Determine the intervals where the following functions are concave up and concave down and locate any inflection points.

(a) \(f(x) = x^3 + x\)
(b) $f(x) = 64x - x^4$

**Graphing Polynomial Functions:**

1. Find the $x-$ and $y$-intercepts from the original function.

2. Find the following information from the first derivative:
   
   (a) Critical values (only need to check where $f'(x) = 0$ since a polynomial)
   
   (b) Intervals of increasing/decreasing
   
   (c) Local extrema

3. Find the following information from the second derivative:

   (a) Concavity
   
   (b) Inflection points

4. Plot all important points (intercepts, local extrema, inflection points) and then sketch a graph of the function corresponding to the information gathered.

5. Check your sketch by graphing the function in an appropriate window on your calculator.
Ex: (#36) Given $f(x) = -4x(x + 2)^3$, analyze $f(x)$ and its first and second derivatives and use the information to sketch a graph of $f(x)$. 
**Def:** A company spends $x$ amount of money in order to increase sales (i.e. advertising). The value of $x$ where the rate of change in sales changes from increasing to decreasing is called the **point of diminishing returns.** This is also the point where the rate of change has a maximum value.

**Ex:**  (#68) A company estimates that it will sell $N(x)$ units of a product after spending $\$x$ thousand on advertising, as given by

$$N(x) = -0.25x^4 + 13x^3 - 180x^2 + 10000 \quad 15 \leq x \leq 24$$

(a) When is the rate of change of sales increasing and when is it decreasing?

(b) What is the point of diminishing returns and the maximum rate of change if sales?