HW1. Solution to Part of Problems.

30. 14. (a) \( f: [0, 1] \rightarrow [0, 2] \)
    \[ f(x) = 2x \]
   (b) \( f: [1, 3] \rightarrow [5, 25] \)
    \[ f(x) = 10x - 5 \]
   (c) \( f: [a, b] \rightarrow [c, d] \)
    \[ f(x) = \frac{d-c}{b-a} x + \frac{c-b}{b-a} \]

Part (a,b) are special cases of (c). Just need to verify for (c) that \( f \) maps \([a, b]\) onto \([c, d]\) and is 1-1.

\( f \) is linear, so it is monotone. Also note \( f \) is continuous.

Easy to check that \( f(a) = c \), \( f(b) = d \). So \( f \) is onto.

If \( f(x) = f(y) \Rightarrow \frac{d-c}{b-a} x + \frac{c-b}{b-a} y \Rightarrow x = y \). So \( f \) is 1-1.

15. Let \( f: (0, 1) \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2}) \) by \( f(x) = \tan(x) \).

By 14, \( f \) is a bijection.

Let \( f_2: (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow IR \) by \( f_2(x) = \tan(x) \)

By Calculus, \( f_2 \) is a bijection.

Hence \( f_2 \circ f: (0, 1) \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2}) \) by \( g(x) = \tan(\tan(x)) \) is a bijection from \((0, 1)\) to \( IR \). This proves \( \text{card}(0,1) = \text{card}(IR) \).

17. If \( |A| = s \times m \), then \( |P(A)| = 2^s \).

Let \( A = \{ a_1, a_2, \ldots, a_s \} \).

Define a map \( f: P(A) \rightarrow \{ 0, 1 \}^s \) by letting

\[
    f(T) = (\xi_1, \xi_2, \ldots, \xi_s)
\]

where

\[
    \xi_i = \begin{cases} 1 & \text{if } a_i \in T \\ 0 & \text{if } a_i \notin T \end{cases}
\]

Then \( f \) is a bijection from \( P(A) \) to \( \{ 0, 1 \}^s \), and \( |f(A)| = 2^s \).
18. Define a map \( f : B^A \rightarrow \mathcal{P}(A) \) by letting

\[
f(g) = \{ x \mid x \in A, \ g(x) = 1 \}.
\]

(i) \( f \) is 1:1: If \( f(g_1) = f(g_2) \)

then \( \{ x \mid x \in A, \ g_1(x) = 1 \} = \{ x \mid x \in A, \ g_2(x) = 1 \} \)

That is, \( \forall x \), \( g_1(x) = 1 \iff g_2(x) = 1 \)

i.e. \( g_1(x) = 0 \iff g_2(x) = 0 \)

Hence \( g_1 = g_2 \)

(ii) \( f \) is onto. \( \forall \) subset \( T \in \mathcal{P}(A) \),

let \( g_T : A \rightarrow \{0, 1\} \) given by

\[
g_T(x) = \begin{cases} 1 & \text{if} \ x \in T \\ 0 & \text{otherwise} \end{cases}
\]

Then \( f(g_T) = T \).

Hence \( f \) is a bijection and \( \text{Card} \left( \mathcal{P}(A) \right) = \text{Card} \left( B^A \right) \).

19. Assume \( \phi \) is a bijection from \( A \) to \( \mathcal{P}(A) \).

Let \( T = \{ x \mid x \in A, \ x \in \phi(x) \} \). Hence \( T \in \mathcal{P}(A) \).

Since \( \phi \) is a bijection, there is an element \( y \in A \) s.t. \( \phi(y) = T \).

Discuss relation between \( y \) and \( T \):

If \( y \in T = \phi(y) \), by def of \( T \), \( y \notin \phi(y) \). Contradiction.

If \( y \notin T = \phi(y) \), by def of \( T \), \( y \in T \). Contradiction again.

Hence there is so such \( \phi \).

20. (a) \( 3 + \infty = \infty \).

We show that there is a bijection between

\( \mathbb{Z}^+ \) and \( A = \{ -2, -1, 0, 1, 2, 3 \} \).

by the setting \( f : \mathbb{Z}^+ \rightarrow A \) as \( f(x) = x - 3 \).

(b) \( \infty + \infty = \infty \).

Let \( A = \mathbb{Z}^+ \), \( B = \mathbb{Z}^- = \{ -1, -2, -3, \ldots \} \), and \( \mathbb{Z}^* = \mathbb{Z} \setminus \{ 0, \infty \} \).

We show that \( A \cup B \) has a 1:1 correspondence to \( \mathbb{Z}^* \).
Let \( f: \mathbb{Z}^* \to A \cup B \) by setting
\[
\begin{align*}
\forall x \in \mathbb{Z}^* & \Rightarrow f(x) = x \\
\forall x \in \{1\} & \Rightarrow f(x) = -x
\end{align*}
\]

b) \( \aleph_0 \times \aleph_0 = \aleph_0 \)

Let \( A = \{ a_1, a_2, a_3, \ldots \} \) and \( B = \{ b_1, b_2, b_3, \ldots \} \)
with \( \text{card}(A) = \text{card}(B) = \aleph_0 \).

We show that \( \text{card}(A \times B) = \aleph_0 \) as well by listing elements in \( A \times B \) in the following order.

29. \( n R m \in \mathbb{Z} \) if \( nm > 0 \).

Not an equivalence relation since we don't have \( 0 R 0 \).

30. \( x R y \in \mathbb{R} \) if \( x > y \).

Not an equivalence relation since not symmetric.
28. $5 + 8$ in $\mathbb{R}_6$ doesn't make sense since 8 is not in $\mathbb{R}_6$.

36. $f: U_1 \rightarrow Z_7$ where $f(5) = 4$.

$U_7: \begin{array}{ccccccc}
1 & 5 & 5^2 & 5^3 & 5^4 & 5^5 & 5^6 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
Z_7: & 0 & 4 & 1 & 5 & 2 & 6 & 3
\end{array}$

39. $Z_1 \times Z_2 = \mathbb{Z}_1 \{ \cos \theta_1 + i \sin \theta_1 \} \cdot \mathbb{Z}_2 \{ \cos \theta_2 + i \sin \theta_2 \}$

$= \mathbb{Z}_1 \| \mathbb{Z}_2 \| \{ \cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2) \}$

40. a) $e^{i \theta} = \cos \theta + i \sin \theta \Rightarrow$

$e^{i \cdot 2 \theta} = (\cos \theta + i \sin \theta)^3 = \cos^3 \theta + 3 \cos^2 \theta \sin \theta \cdot i + 3 \cos \theta (i \sin \theta)^2$

$= \cos^3 \theta - 3 \cos \theta \sin^2 \theta + i [3 \cos^2 \theta \sin \theta - \sin^3 \theta]$

Hence $\cos 3 \theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$.

b) $\cos \beta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$

$= \cos^3 \theta - 3 \cos \theta (1 - \cos^2 \theta)$

$= 4 \cos^3 \theta - 3 \cos \theta$. 