Chapter 2

PLANE AND SPACE VECTORS

2.1 VECTORS

In chemical engineering one is interested in the concentrations of a chemical species at different levels of an absorption column (see Fig. 2.1.1). To record the concentrations at a certain time, we can write

\[ C = (c_1, c_2, \ldots, c_n) \]

where \( c_k \) is the concentration at the \( k \)th level. We can call \( C \) a concentration vector; it is just a \( 1 \times n \) matrix.

**Definition 2.1.1.** An \( n \)-vector is a \( 1 \times n \) matrix.

Before developing the algebra of \( n \)-vectors in Chap. 3, we study two-vectors and three-vectors because their structure can be visualized geometrically.

**Definition 2.1.2.** A plane vector (or two-vector) is a \( 1 \times 2 \) matrix with real entries. A space vector (or three-vector) is a \( 1 \times 3 \) matrix with real entries. The set of all plane vectors is denoted \( \mathbb{R}^2 \); the set of all space vectors, \( \mathbb{R}^3 \).

The elements of the matrix are called the components of the vector. Two vectors are equal if they are equal as matrices.
Example 1. The following are plane vectors: $(0,1)$, $(\pi, 1)$, $(-7, 2)$. Commas are placed between the components to emphasize the fact that plane vectors are associated with points in the standard $xy$ plane (see Fig. 2.1.2). If we connect the point to the origin with an arrow (see Fig. 2.1.2), we call the arrow a geometric representation of the vector.

Example 2. The following are space vectors: $(1, 1, -1)$, $(2, 0, 1)$, $(-1, 1, 1)$. We associate these vectors with points in the standard $xyz$ coordinate system, as shown in Fig. 2.1.3.

A plane vector $(a, b)$ is also associated with any arrow in the standard $xy$ plane which begins at a point $(x_0, y_0)$ and ends at $(x_0 + a, y_0 + b)$. In Fig. 2.1.4 we have drawn several arrows associated with the vector $(1,2)$.

In general, when we write a vector $(a, b)$, we assume that the initial point of the associated vector is $(0,0)$ unless specified otherwise.

Example 3. Find the vector associated with the arrow with initial point $(-2, 4)$ and terminal point $(-7, 3)$.

Solution Now $(-7, 3)$ must be equal to $(x_0 + a, y_0 + b)$. Since $(x_0, y_0) = (-2, 4)$, we have $(-7, 3) = (-2 + a, 4 + b)$. Thus $a = -5$, $b = -1$, so $(a, b) = (-5, -1)$. The easy way to solve this problem is to “subtract points”: terminal point – initial point. In this way (just subtracting matrices)

$$(a, b) = (-7, 3) - (-2, 4) = (-5, -1)$$

See Fig. 2.1.5.

For space vectors we have similar methods and interpretations in the $xyz$ coordinate system.

Example 4. Sketch the arrow corresponding to the vector $(1, -1, 2)$ with initial point $(0,0,0)$ and with initial point $(1, 2, -1)$.

Solution We draw the standard $x$, $y$, and $z$ axes, locate $(0,0,0)$ and $(1, -1, 2)$, and draw an arrow from $(0,0,0)$ to $(1, -1, 2)$. See Fig. 2.1.6. For the second case we locate the point $(1, 2, -1)$ as the initial point. Let the terminal point be $(a, b, c)$; then the vector $(1, -1, 2)$ satisfies

$$(1, -1, 2) = (a, b, c) - (1, 2, -1)$$

This means that the terminal point is $(2, 2, 1)$. 
The length of the arrow representing a vector is defined to be the **length**, or **magnitude**, of the vector. Considering Fig. 2.1.4 again and using Pythagoras’ theorem for right triangles, we find that the length of $(1,2)$ is $\sqrt{1^2 + 2^2} = \sqrt{5}$. So we have the following definition.

**Definition 2.1.3.** The **length** (or **magnitude**) of the vector $\mathbf{A} = (a, b)$ is denoted $|\mathbf{A}|$ or $|(a, b)|$ and is defined by

$$|\mathbf{A}| = \sqrt{a^2 + b^2}$$

For three-vectors

$$|(a, b, c)| = \sqrt{a^2 + b^2 + c^2}$$

**Example 5.** Calculate $|(0, 0)|, |(1, -1)|, |(3, 4)|, |(0, 1, 0)|$, and $|(\sqrt{11}, 3, 4)|$.

**Solution**

$$|(0, 0)| = \sqrt{0^2 + 0^2} = \sqrt{0} = 0$$

$$|(1, -1)| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$|(3, 4)| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$|(0, 1, 0)| = \sqrt{0^2 + 1^2 + 0^2} = \sqrt{1} = 1$$

$$|(\sqrt{11}, 3, 4)| = \sqrt{(\sqrt{11})^2 + 3^2 + 4^2} = \sqrt{36} = 6$$

**Definition 2.1.4.** The **sum** of vectors $(a, b)$ and $(c, d)$ is defined as

$$(a, b) + (c, d) = (a + c, b + d)$$

The **sum** of vectors $(a, b, c)$ and $(d, e, f)$ is defined as

$$(a, b, c) + (d, e, f) = (a + d, b + e, c + f)$$

**Example 6.** Let $\mathbf{A} = (1, 0, -1), \mathbf{B} = (3, -4, 2), \mathbf{C} = (\pi, 0, 4), \mathbf{D} = (0, 0, 0)$, and $\mathbf{E} = (-1, 0, 1)$. Calculate $\mathbf{A} + \mathbf{B}, \mathbf{B} + \mathbf{D}, \mathbf{A} + \mathbf{B} + \mathbf{C}$, and $\mathbf{A} + \mathbf{E}$.

**Solution**

$$\mathbf{A} + \mathbf{B} = (1 + 3, 0 + -4, -1 + 2) = (4, -4, 1)$$

$$\mathbf{B} + \mathbf{D} = (3 + 0, -4 + 0, 2 + 0) = (3, -4, 2) = \mathbf{B}$$

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = (1 + 3 + \pi, 0 + (-4) + 0, -1 + 2 + 4) = (4 + \pi, -4, 5)$$

$$\mathbf{A} + \mathbf{E} = (1 + -1, 0 + 0, -1 + 1) = (0, 0, 0)$$
The definition of addition corresponds to the “addition” of forces represented by the vectors. In Fig. 2.1.7 we show the representation of two forces acting on a point at the origin; the sum of the two vectors is called the resultant force. The length of each vector corresponds to the magnitude of the force, and the direction to the direction of application of the force. The resultant vector is just the diagonal (having initial point in common with the two forces) of the parallelogram generated by the two forces.

Because vectors represent forces, some other concepts regarding forces translate to operations with vectors. For instance, if we will “twice as hard” on an object, we think of doubling the force we are applying. If the original force is \((a, b)\), the doubled force is \((2a, 2b)\). This leads to the definition of scalar multiple.

**Definition 2.1.5.** Let \(r\) be a real number and \((a, b)\) be a vector. The scalar\(^1\) multiple \(r(a, b)\) is defined as

\[
r(a, b) = (ra, rb)
\]

The reader should write a definition for a scalar multiple of \((a, b, c)\). Note that this multiplication is the same as multiplication of a matrix by a scalar.

The idea of reversing a force or changing it to the opposite direction leads to a definition of the negative of a vector.

**Definition 2.1.6.** The negative of a vector \(\mathbf{V}\) is written \(-\mathbf{V}\) and is defined as \((-1)\mathbf{V}\). The difference \(\mathbf{A} - \mathbf{B}\) of two vectors \(\mathbf{A}\) and \(\mathbf{B}\) is defined by

\[
\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B}
\]

**Example 7.** Let \(\mathbf{A} = (1, -1)\) and \(\mathbf{B} = (2, 4)\). Calculate \(4\mathbf{A} + \mathbf{B}\), \(\mathbf{A} - \mathbf{B}\), and \(\mathbf{A} - \mathbf{A}\).

**Solution**

\[
4\mathbf{A} + \mathbf{B} = 4(1, -1) + (2, 4) = (4, -4) + (2, 4) = (6, 0)
\]

\[
\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B} = (1, -1) + (-2, -4) = (-1, -5)
\]

\(^1\)In this chapter, scalar is restricted to mean “real number.”
or more simply

\[(1, -1) - (2, 4) = (1 - 2, -1 - 4) = (-1, -5)\]

Finally,

\[\mathbf{A} - \mathbf{A} = (1, -1) - (1, -1) = (1 - 1, -1 - (-1)) = (0, 0)\]

**Example 8.** Represent geometrically the vector difference \(\mathbf{A} - \mathbf{B}\).

**Solution**  Now \(\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})\). In Fig. 2.1.8 we show the resultant of \(\mathbf{A}\) and \(-\mathbf{B}\). Since the vector \(\mathbf{A} - \mathbf{B}\) can be moved as long as we do not change direction or length, \(\mathbf{A} - \mathbf{B}\) can be found geometrically by drawing an arrow from the end of \(\mathbf{B}\) to the end of \(\mathbf{A}\) (\(\mathbf{A}\) and \(\mathbf{B}\) have the same initial point). See Fig. 2.1.9.

Geometric interpretations of vector addition can be used to study the result of several forces acting on one body. These kinds of problems are considered in the study of mechanics, and we do not pursue them here.

So far, we have natural ways to add and subtract vectors as well as a natural way to multiply a vector by a scalar. Since vectors are just matrices, and the operations are the matrix operations all our properties from Chap. 1 hold for vectors. We repeat these below.

**Theorem 2.1.1.** Let \(\mathbf{U}, \mathbf{V}\) and \(\mathbf{W}\) all be plane vectors or all space vectors. Let \(\mathbf{0}\) be \((0,0)\) or \((0,0,0)\), respectively, depending on the size of \(\mathbf{U}, \mathbf{V}\), and \(\mathbf{W}\). Let \(r\) and \(s\) be real numbers. Then we have

\[
\begin{align*}
\mathbf{U} + \mathbf{V} &= \mathbf{V} + \mathbf{U} \\
\mathbf{U} + (\mathbf{V} + \mathbf{W}) &= (\mathbf{U} + \mathbf{V}) + \mathbf{W} \\
\mathbf{U} + \mathbf{0} &= \mathbf{0} + \mathbf{U} = \mathbf{U} \\
\mathbf{U} + (-\mathbf{U}) &= \mathbf{0} \\
r(s\mathbf{U}) &= (rs)\mathbf{U} = s(r\mathbf{U}) \\
(r + s)\mathbf{U} &= r\mathbf{U} + s\mathbf{U} \\
r(\mathbf{U} + \mathbf{V}) &= r\mathbf{U} + r\mathbf{V} \\
0\mathbf{U} &= \mathbf{0} \\
1\mathbf{U} &= \mathbf{U}
\end{align*}
\]
The Vectors $\mathbf{i,j,k}$ A common way of representing three-vectors in engineering is to use the vectors
\[
\mathbf{i} = (1, 0, 0) \\
\mathbf{j} = (0, 1, 0) \\
\mathbf{k} = (0, 0, 1)
\]
In particular, using the results of matrix algebra, we have
\[
(a, b, c) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}
\]
For the two-vector case, $\mathbf{i}$ and $\mathbf{j}$ are defined as $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$; thus $(a, b) = a\mathbf{i} + b\mathbf{j}$. In either case, the vectors $\mathbf{i}, \mathbf{j},$ and $\mathbf{k}$ are called standard basis vectors (the reason for this terminology is seen in Chap. 3), and the coefficients of $\mathbf{i}, \mathbf{j},$ and $\mathbf{k}$ are called the standard coordinates of the vector.

**Definition 2.1.7.** Let $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ be a three-vector. The column matrix
\[
\mathbf{v}_s = \begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
\]
is called the standard coordinate matrix of $\mathbf{v}$. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is a two-vector, its standard coordinate matrix is
\[
\mathbf{v}_s = \begin{pmatrix}
a \\
b
\end{pmatrix}
\]

**Example 9.** Show that if $\mathbf{v}$ and $\mathbf{w}$ are the three-vectors $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$, then the standard coordinate matrix of $\mathbf{v} + \mathbf{w}$ is the sum of the standard coordinate matrices of $\mathbf{v}$ and $\mathbf{w}$. That is, show that
\[
(\mathbf{v} + \mathbf{w})_s = \mathbf{v}_s + \mathbf{w}_s
\]

**Solution**
\[
\mathbf{v}_s = \begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix} \quad \mathbf{w}_s = \begin{pmatrix}
w_1 \\
w_2 \\
w_3
\end{pmatrix}
\]
The vector $\mathbf{v} + \mathbf{w} = (v_1 + w_1)\mathbf{i} + (v_2 + w_2)\mathbf{j} + (v_3 + w_3)\mathbf{k}$, so that
\[
(\mathbf{v} + \mathbf{w})_s = \begin{pmatrix}
v_1 + w_1 \\
v_2 + w_2 \\
v_3 + w_3
\end{pmatrix} = \mathbf{v}_s + \mathbf{w}_s
\]
Example 10. Show that multiplication of the coordinate matrix of a two-vector \( v \) by

\[
A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}
\]

results in the standard coordinate matrix of \( 3v \).

Solution Let \( v = ai + bj \). Then

\[
X = \begin{pmatrix} a \\ b \end{pmatrix}
\]

is the standard coordinate matrix of \( v \). Now \( 3v = 3ai + 3bj \) and

\[
AX = \begin{pmatrix} 3a \\ 3b \end{pmatrix}
\]

which is the standard coordinate matrix of \( 3v \).

Example 11. Let \( A \) be a \( 2 \times 2 \) matrix and \( v \) a two-vector. Show that \( Av_s = (vA^T)_s \).

Solution For any vector \( w \) with \( w = (a, b) \), we have

\[
w_s = \begin{pmatrix} a \\ b \end{pmatrix}
\]

so that \( w_s = w^T \). Therefore \( (vA^T)_s = (vA^T)^T = A^T v^T = Av^T = Av_s \).

Introduction of standard coordinate matrices allows us to “stand \((a, b, c)\) up” as

\[
\begin{pmatrix} a \\ b \\ c \end{pmatrix}
\]

The purpose is to allow multiplication of a vector on the left by a matrix. If \( A \) is \( 3 \times 3 \) and \( v = (a, b, c) \), then \( Av \) makes no sense, but \( Av_s \) does. In Sec. 2.3 we will use this multiplication. Note that \( v_s \neq v \) but \( v_s \) represents \( v \) in a natural way.
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Problems 2.1

1. Sketch arrows with initial point (0,0) corresponding to the following vectors:
   (a) (1,2)    (b) (−1, −3)    (c) (−1, 2)    (d) (π, −1)

2. Sketch the vectors in Prob. 1 so that their initial point is
   (a) (1,2)    (b) (−3, −5)

3. A pair of points is given. Find the vector beginning at the first point and ending at the second.
   (a) (1, −1), (3, 5)     (b) (3, 2), (−3, −7)
   (c) (−1, −6), (2, −8)  (d) (1, 1), (7, −8)
   (e) (1, −1, 2), (0, −2, −6)     (f) (−3, 0, 1), (1, 1, 2)

4. Sketch arrows corresponding to the following vectors:  (a) (1, −2, 0),
   (b) (−2, −4, 3), and (c) (4,4,7).

5. Calculate the length of the vectors in Prob. 1.

6. Calculate the length of the vectors in Prob. 4.

7. What value should \( k \) have to make \( |(k, 3k, 4k)| = 1 \)?

8. Let \( \mathbf{A} = (1, −1, 2), \mathbf{B} = (3, 0, 4), \mathbf{C} = (4,2), \mathbf{D} = (−1, 1), \mathbf{i} = (1, 0), \) and \( \mathbf{j} = (0, 1) \). Calculate
   (a) \( \mathbf{A} + \mathbf{B} \)    (b) \( \mathbf{A} − \mathbf{B} \)    (c) \( 2\mathbf{C} − 3\mathbf{D} \)
   (d) \( 4\mathbf{i} + 2\mathbf{j} \)    (e) \( |\mathbf{i}|, |\mathbf{j}|, |\mathbf{i} + \mathbf{j}| \)

9. Let \( \mathbf{A} = (1, 2) \) and \( \mathbf{B} = (−3, 1) \). Calculate and sketch the given vector.
   (a) \( \mathbf{A} + \mathbf{B} \)    (b) \( \mathbf{A} − \mathbf{B} \)    (c) \( \mathbf{B} − \mathbf{A} \)    (d) \( 2\mathbf{A} + 3\mathbf{B} \).

10. Let \( \mathbf{A} = (1, 2) \) and \( \mathbf{B} = (−k^2, k^2 − k) \). Are there any real values of \( k \)
    which make \( \mathbf{A} + \mathbf{B} = 0? \)

11. Show that if \( \mathbf{A} ≠ 0 \), then the vector \( (1/|\mathbf{A}|)\mathbf{A} \) has length 1.

12. Show by example that \( |\mathbf{A} + \mathbf{B}| \) is not necessarily equal to \( |\mathbf{A}| + |\mathbf{B}| \).
    Using the geometric diagram for vector addition, reason that \( |\mathbf{A} + \mathbf{B}| ≤ |\mathbf{A}| + |\mathbf{B}| \).
13. Show that if \( \mathbf{v} \) is a two- or three-vector and \( r \) is a real number, then \( (r\mathbf{v})_s = r\mathbf{v}_s \).

14. Show that if \( \mathbf{v} \) and \( \mathbf{w} \) are three-vectors, then \( (\mathbf{v} - \mathbf{w})_s = \mathbf{v}_s - \mathbf{w}_s \).

15. Show that if \( A \) is a \( 3 \times 3 \) matrix and \( \mathbf{v} \) is a three-vector, then \( A\mathbf{v} = (\mathbf{v}A^T)_s \).

### 2.2 THE ANGLE BETWEEN VECTORS; PROJECTIONS

One of the most important problems in the analysis of vectors is the **angle problem**: Given two vectors \( \mathbf{A} \) and \( \mathbf{B} \), find the angle \( \theta \), \( 0 \leq \theta \leq \pi \), between \( \mathbf{A} \) and \( \mathbf{B} \). See Fig. 2.2.1.

If we can solve this problem, then we know whether \( \mathbf{A} \) is parallel to \( \mathbf{B} \) (\( \theta \) is 0 or \( \pi \)) or \( \mathbf{A} \) is perpendicular to \( \mathbf{B} \) (\( \theta = \pi/2 \)).

The solution to this problem for plane vectors can be found by using the law of cosines: For a triangle with sides of length \( a, b \), and \( c \) as shown

![Figure](image)

we have

\[
c^2 = a^2 + b^2 - 2ab \cos \theta \tag{2.2.1}
\]

From Eq. (2.2.1) we find

\[
\cos \theta = \frac{a^2 + b^2 - c^2}{2ab} \tag{2.2.2}
\]

Therefore in any triangle with lengths of sides given, we can find the cosine of any of the angles. And this is just as good as finding \( \theta \)—especially for the purposes of determining whether \( \theta \) equals 0, \( \pi/2 \), or \( \pi \).

To solve the angle problem, we consider the triangle formed by \( \mathbf{A}, \mathbf{B} \), and \( \mathbf{A} - \mathbf{B} \) (see Fig. 2.2.2) and use the law of cosines. We have from (2.2.2)

\[
\cos \theta = \frac{|\mathbf{A}|^2 + |\mathbf{B}|^2 - |\mathbf{A} - \mathbf{B}|^2}{2|\mathbf{A}||\mathbf{B}|} \tag{2.2.3}
\]
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Suppose now that \( \mathbf{A} = (a_1, a_2) \) and \( \mathbf{B} = (b_1, b_2) \). Then \( \mathbf{A} - \mathbf{B} = (a_1 - b_1, a_2 - b_2) \) and

\[
|\mathbf{A}|^2 + |\mathbf{B}|^2 = |\mathbf{A} - \mathbf{B}|^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - [(a_1 - b_1)^2 + (a_2 - b_2)^2]
= a_1^2 + a_2^2 + b_1^2 + b_2^2
- (a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2)
= 2(a_1b_1 + a_2b_2)
\]

This means that

\[
\cos \theta = \frac{a_1b_1 + a_2b_2}{|\mathbf{A}| \ |\mathbf{B}|} \quad (2.2.4)
\]

and the angle problem is solved for two-vectors.

**Example 1.** Find the cosine of the angle between \( \mathbf{A} = (3, 4) \) and \( \mathbf{B} = (5, 12) \).

**Solution** Since \( a_1 = 3, \ a_2 = 4, \ b_1 = 5, \) and \( b_2 = 12, \) we have

\[
\cos \theta = \frac{3 \cdot 5 + 4 \cdot 12}{\sqrt{3^2 + 4^2} \sqrt{5^2 + 12^2}} = \frac{63}{5 \cdot 13} = \frac{63}{65}
\]

**Example 2.** Find the cosine of the angle between \( \mathbf{A} = (1, 1) \) and \( \mathbf{B} = (-2, 2) \). Sketch the vectors. Are they perpendicular?

**Solution** We have

\[
\cos \theta = \frac{1(-2) + (+1)(2)}{\sqrt{1^2 + (-1)^2} \sqrt{(-2)^2 + 2^2}} = 0
\]

The vectors are sketched in Fig. 2.2.3. Because \( \cos \theta = 0, \ \theta = \pi/2, \) so the vectors are perpendicular.

If we look at Eq. (2.2.4) again, we see that the numerator of the right-hand side, \( a_1b_1 + a_2b_2, \) is actually (in terms of matrix multiplication)

\[
\mathbf{A}\mathbf{B}^T = (a_1a_2) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1b_1 + a_2b_2
\]

Therefore Eq. (2.2.4) can be written

\[
\cos \theta = \frac{\mathbf{A}\mathbf{B}^T}{|\mathbf{A}| \ |\mathbf{B}|} \quad \text{solution to angle problem} \quad (2.2.5)
\]

It turns out that this formula also works for three-vectors.
Example 3. Find the cosine of the angle between \( \mathbf{A} = (1, -1, 4) \) and \( \mathbf{B} = (4, 0, -1) \).

Solution By 2.2.5

\[
\cos \theta = \frac{\mathbf{A}^T \mathbf{B}}{||\mathbf{A}|| ||\mathbf{B}||} = \frac{4 + 0 - 4}{\sqrt{18} \sqrt{17}} = 0
\]

Hence, these vectors are perpendicular.

Since the matrix product \( \mathbf{A}^T \mathbf{B} \) is used to solve the angle problem for vectors, we give it a special name, the dot product.

Definition 2.2.1. Given two plane or space vectors \( \mathbf{A} \) and \( \mathbf{B} \), the dot product \( \mathbf{A} \cdot \mathbf{B} \) is defined by \( \mathbf{A} \cdot \mathbf{B} = \mathbf{A}^T \mathbf{B} \). Because \( \mathbf{A} \cdot \mathbf{B} \) is a real number, \( (\mathbf{A} \cdot \mathbf{B})^T = \mathbf{A} \cdot \mathbf{B} \). Therefore \( \mathbf{B} \cdot \mathbf{A} = \mathbf{B}^T \mathbf{A} = (\mathbf{A}^T \mathbf{B})^T = (\mathbf{A} \cdot \mathbf{B})^T = \mathbf{A} \cdot \mathbf{B} \).

This means that the dot product is commutative.

Example 4. Let \( \mathbf{A} = (-1, 4) \), \( \mathbf{B} = (2, 3) \), \( \mathbf{C} = (7, 0, 2) \), and \( \mathbf{D} = (-1, 1, 1) \). Calculate, if possible, \( \mathbf{A} \cdot \mathbf{B} \), \( \mathbf{B} \cdot \mathbf{C} \), and \( \mathbf{C} \cdot \mathbf{D} \).

Solution Now

\[
\mathbf{A} \cdot \mathbf{B} = (-1, 4) \left( \begin{array}{c} 2 \\ 3 \end{array} \right) = -2 + 12 = 10
\]

and \( \mathbf{B} \cdot \mathbf{C} \) cannot be calculated since \( \mathbf{B} \) is \( 1 \times 3 \) and \( \mathbf{C}^T \) is \( 2 \times 1 \). So \( \mathbf{B} \mathbf{C}^T \) does not exist.

\[
\mathbf{C} \cdot \mathbf{D} = \mathbf{C}^T \mathbf{D} = (7, 0, 2) \left( \begin{array}{c} -1 \\ 1 \\ 1 \end{array} \right) = -7 + 2 = -5
\]

We can state several properties of the dot product now.

Theorem 2.2.1. Let \( \mathbf{A} \), \( \mathbf{B} \), and \( \mathbf{C} \) be vectors of the same order. Let \( r \) and \( s \) be real numbers.

(a) \( \mathbf{A} \cdot \mathbf{B} = ||\mathbf{A}|| ||\mathbf{B}|| \cos \theta \), where \( \theta \) is the angle between \( \mathbf{A} \) and \( \mathbf{B} \).

(b) \( \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \)

(c) \( (\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C} \)
(d) \( r(A \cdot B) = (rA) \cdot B = A \cdot (rB) \)

(e) \( (rA) \cdot (sB) = (rs)(A \cdot B) \)

(f) \( \sqrt{A \cdot A} = |A| \)

(g) \( A \cdot A = 0 \) if and only if \( A = 0 \) (the zero vector).

(h) Let \( |A| \neq 0 \) and \( |B| \neq 0 \). Then \( A \cdot B = 0 \) if and only if \( A \perp B \).

(i) \( |A \cdot B| \leq |A| \cdot |B| \) (Cauchy-Schwarz inequality)

(j) \( |rA| = |r| \cdot |A| \)

(k) \( |A \cdot B| \leq |A| + |B| \)

Proof. (a) This is just a restatement of Eq. (2.2.5).

(b) We have \( A \cdot (B + C) = A(B + C)^T = AB^T + AC^T \), where the last equality follows from the properties of the transpose and matrix multiplication. The last term is just \( A \cdot B + A \cdot C \) by definition of dot product.

(c) (d), and (e) follows from matrix properties and are proved in a fashion similar to (b).

(f) Consider the case of three-vectors. Let \( A = (a_1, a_2, a_3) \). Then \( \sqrt{A \cdot A} = \sqrt{a_1^2 + a_2^2 + a_3^2} \) which is just \( |A| \). For two-vectors the argument is the same.

(g) First we prove that if \( A = 0 \), then \( A \cdot A = 0 \). If \( A = 0 = (0, 0, 0) \), then \( A \cdot A = 0^2 + 0^2 + 0^2 = 0 \). Now we show that if \( A \cdot A = 0 \), then \( A \) must be the zero vector. Let \( A = (a_1, a_2, a_3) \), so that \( A \cdot A = a_1^2 + a_2^2 + a_3^2 \). Since \( a_1^2, a_2^2 \) and \( a_3^2 \) are all greater than or equal to zero, the only way the sum of these can be zero is if each term is zero. That is, we must have \( a_1^2 = a_2^2 = a_3^2 = 0 \). So \( a_1 = a_2 = a_3 = 0 \) and \( A = 0 \).

(h) From (a), \( A \cdot B = |A| \cdot |B| \cos \theta \). Since \( |A| \neq 0 \) and \( |B| \neq 0 \), the only way \( A \cdot B \) can equal zero is if \( \cos \theta = 0 \). That is, \( \theta = \pi/2 \). Therefore, \( A \) is perpendicular to \( B \).

(i) From (a) \( |A \cdot B| = |A| \cdot |B| \cdot | \cos \theta | \). Since \( | \cos \theta | \leq 1 \), \( |A \cdot B| \leq |A| \cdot |B| \).

Proofs of (j) and (k) are left to the problems.
2.2. THE ANGLE BETWEEN VECTORS; PROJECTIONS

Projections are an important geometric idea which we can now discuss by using the dot product. Geometrically, the projection of a vector \( \mathbf{B} \) on a vector \( \mathbf{A} \) is shown in Fig. 2.2.4. Roughly speaking, the projection of \( \mathbf{B} \) on \( \mathbf{A} \) is the shadow which \( \mathbf{B} \) casts on \( \mathbf{A} \) due to light rays which hit \( \mathbf{A} \), the light rays being perpendicular to \( \mathbf{A} \). In mechanics, the projection is the component of the force \( \mathbf{B} \) in the direction of \( \mathbf{A} \), and \( \mathbf{B} \cdot \mathbf{A} \) is the work done by the force \( \mathbf{B} \) in the direction of \( \mathbf{A} \).

We can determine the vector projection of \( \mathbf{B} \) on \( \mathbf{A} \) (written \( \mathbf{B}_{\text{proj}} \mathbf{A} \)) by finding its length; its direction is the same as \( \mathbf{A} \). Once we find \( |\mathbf{B}_{\text{proj}} \mathbf{A}| \), we can multiply this by \( (1/|\mathbf{A}|) \mathbf{A} \) (the unit vector in the direction of \( \mathbf{A} \)) to get \( \mathbf{B}_{\text{proj}} \mathbf{A} \). We find \( |\mathbf{B}_{\text{proj}} \mathbf{A}| \) by using trigonometry. From Fig. 2.2.5 we see that
\[
|\mathbf{B}_{\text{proj}} \mathbf{A}| = |\mathbf{B}| \cos \theta,
\]
where \( \theta \) is the angle between \( \mathbf{A} \) and \( \mathbf{B} \). Therefore, since
\[
\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}| |\mathbf{B}|} = \cos \theta,
\]
we know that
\[
|\mathbf{B}_{\text{proj}} \mathbf{A}| = |\mathbf{B}| \cos \theta = |\mathbf{B}| \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}| |\mathbf{B}|} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}|}
\]
and so
\[
\mathbf{B}_{\text{proj}} \mathbf{A} = |\mathbf{B}_{\text{proj}} \mathbf{A}| \frac{1}{|\mathbf{A}|} \mathbf{A} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}|^2} \mathbf{A}
\]
Sometimes \( |\mathbf{B}_{\text{proj}} \mathbf{A}| \) is called the scalar projection of \( \mathbf{B} \) on \( \mathbf{A} \).

Example 5. Let \( \mathbf{A} = (1, 0, 1) \) and \( \mathbf{B} = (1, -1, 2) \). Find \( \mathbf{B}_{\text{proj}} \mathbf{A} \).

Solution
\[
\mathbf{B}_{\text{proj}} \mathbf{A} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}|^2} \mathbf{A} = \frac{3}{2} (1, 0, 1) + \left( \frac{3}{2}, 0, \frac{3}{2} \right)
\]

A vector product which is important in three-dimensional mechanics is the cross product. This product is specific to three-vectors and does not generalize in a natural way to higher dimensions. Because of this, our treatment of the cross product is brief.

Definition 2.2.2. Let \( \mathbf{A} = (a_1, a_2, a_3) \) and \( \mathbf{B} = (b_1, b_2, b_3) \). Then the cross product of \( \mathbf{A} \) and \( \mathbf{B} \), written \( \mathbf{A} \times \mathbf{B} \), is a vector defined by
\[
\mathbf{A} \times \mathbf{B} = (a_2b_3 - b_2a_3, -a_1b_3 + b_1a_3, a_1b_2 - a_2b_1)
\]
Example 6. Calculate \( \mathbf{A} \times \mathbf{B} \) for \( \mathbf{A} = (1, -1, 2) \) and \( \mathbf{B} = (2, 2, 7) \).

Solution One way to do this is to form a symbolic determinant

\[
\begin{bmatrix}
1 & 2 & 3 \\
1 & -1 & 2 \\
2 & 2 & 7
\end{bmatrix}
\]

component of \( \mathbf{A} \times \mathbf{B} \)

and to expand by the first row:

\[
\mathbf{A} \times \mathbf{B} = \mathbf{i} \det \begin{pmatrix} -1 & 2 \\ 2 & 7 \end{pmatrix} - \mathbf{j} \det \begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix} + \mathbf{k} \det \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} = (-11, -3, 4)
\]

Notice that \( \mathbf{A} \times \mathbf{B} \) is perpendicular to both \( \mathbf{A} \) and \( \mathbf{B} \) since

\[
(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{A} = (-11, -3, 4) \cdot (1, -1, 2) = -11 + 3 + 8 = 0
\]

\[
(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{B} = (-11, -3, 4) \cdot (2, 2, 7) = -22 - 6 + 28 = 0
\]

In fact, this is always true.

Theorem 2.2.2. Let \( \mathbf{A} \) and \( \mathbf{B} \) be nonzero vectors. Then \( (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{A} = 0 = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{B} \).

Proof. For \( (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{A} \) we have

\[
(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{A} = (a_2b_3 - b_2a_3, -a_1b_3 + b_1a_3, a_1b_2 - a_2b_1) \cdot (a_1, a_2, a_3)
\]

\[
= a_1a_2b_3 - a_1b_2a_3 - a_2a_1b_3 + a_2b_1a_3
\]

\[
+ a_3a_1b_2 - a_3b_1a_3
\]

\[
= 0
\]

Geometrically \( \mathbf{A} \times \mathbf{B} \) is the vector of length \( |\mathbf{A}| \, |\mathbf{B}| \sin \theta \), perpendicular to both \( \mathbf{A} \) and \( \mathbf{B} \) and pointing in the direction dictated by the right-hand rule: Using the right hand, curve your fingers from \( \mathbf{A} \) to \( \mathbf{B} \); \( \mathbf{A} \times \mathbf{B} \) points in the direction of the thumb. (See Fig. 2.2.6.) This can be shown rigorously, but we omit the proof.

From the geometric meaning of \( \mathbf{A} \times \mathbf{B} \) it is not hard to see that \( \mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B}) \) and \( \mathbf{A} \times \mathbf{B} = 0 \) if \( \sin \theta = 0 \), that is, \( \mathbf{A} \times \mathbf{B} = 0 \) if \( \mathbf{A} \) is parallel to \( \mathbf{B} \) (the directions can be opposite). \( \square \)

Problems 2.2

1. Calculate \( \mathbf{A} \cdot \mathbf{B} \).
2.2. **THE ANGLE BETWEEN VECTORS; PROJECTIONS**

(a) \( \mathbf{A} = (1, 2), \mathbf{B} = (3, 7) \)
(b) \( \mathbf{A} = (-5, 2), \mathbf{B} = (3, 6) \)
(c) \( \mathbf{A} = (5, 4, 0), \mathbf{B} = (0, 1, 3) \)
(d) \( \mathbf{A} = (0, 1, -1), \mathbf{B} = (6, 6, 7) \)

2. Find the angle between \( \mathbf{A} \) and \( \mathbf{B} \).

(a) \( \mathbf{A} = (5, 0), \mathbf{B} = (2, 2) \)
(b) \( \mathbf{A} = (2, -2), \mathbf{B} = (-1, 1) \)
(c) \( \mathbf{A} = (3, 0, 4), \mathbf{B} = (-6, 0, -8) \)

3. Find the cosine of the angle between the vectors in Prob. 1.

4. For the vectors in Prob. 1, verify the Cauchy-Schwarz inequality \( |\mathbf{A} \cdot \mathbf{B}| \leq |\mathbf{A}| \cdot |\mathbf{B}| \).

5. Find the projection \( \mathbf{B} \) on \( \mathbf{A} \) for the vectors in Prob. 1.

6. Calculate \( \mathbf{A} \times \mathbf{B} \) for the vectors in Prob. 1c and d.

7. Show that for the set of vectors \( \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \) any vector from the set is perpendicular to all others in the set.

8. (a) Find a vector perpendicular to \( (1, 2) \).
(b) Find a vector of length 1 unit perpendicular to \( (1, 2) \).
(c) How many correct answers are there to part (b)?

9. Find the angle between the diagonal of a cube and one of its edges.

10. Let \( \mathbf{A} \neq \mathbf{0}, \mathbf{B} \neq \mathbf{0}, \text{ and } \mathbf{A} \neq \pm \mathbf{B} \). Show that the angle between \( \mathbf{A} + \mathbf{B} \) and \( \mathbf{B} \) is less than the angle between \( \mathbf{A} \) and \( \mathbf{B} \). **Hint:** The cosines of the angles are

\[
\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}| \cdot |\mathbf{B}|} \quad \text{and} \quad \frac{(\mathbf{A} + \mathbf{B}) \cdot \mathbf{B}}{|\mathbf{A} + \mathbf{B}| \cdot |\mathbf{B}|}
\]

since \( 0 \leq \theta \leq 180^\circ \) of \( \theta_1 < \theta_2 \), \( \cos \theta_1 > \cos \theta_2 \). Show that

\[
\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}| \cdot |\mathbf{B}|} < \frac{(\mathbf{A} + \mathbf{B}) \cdot \mathbf{B}}{|\mathbf{A} + \mathbf{B}| \cdot |\mathbf{B}|}
\]

by using \( |\mathbf{A} + \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}| \) and \( \mathbf{A} \cdot \mathbf{B} \leq |\mathbf{A}| \cdot |\mathbf{B}| \).

11. **Parallelogram identity.** Consider the parallelogram
FIGURE

By the law of cosines

$$|\mathbf{A} - \mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}|\cos \theta$$
$$|\mathbf{A} + \mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}|\cos(180^\circ - \theta)$$

Use this to show that

(a) $\mathbf{A} \cdot \mathbf{B} = \frac{1}{2}|\mathbf{A} + \mathbf{B}|^2 - \frac{1}{2}|\mathbf{A} - \mathbf{B}|^2$

(b) $|\mathbf{A}|^2 + |\mathbf{B}|^2 = \frac{1}{2}(|\mathbf{A} + \mathbf{B}|^2 + |\mathbf{A} - \mathbf{B}|^2)$

In words, (b) says that the sum of the squares of the sides of a parallelogram is the average of the squares of the lengths of the diagonals.

12. Show that $\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B})$, using Definition 2.2.2.

13. Show that $\mathbf{B} - \mathbf{B}_{\text{proj } \mathbf{A}}$ is perpendicular to $\mathbf{A}$.

2.3 MATRICES AS TRANSFORMERS OF SPACE

Analysis of functions is the lifeblood of applied mathematics. In calculus, functions with various domains and ranges were studied—a summary is given in Table 2.3.1. In linear algebra we analyze certain functions with domains and ranges which are like $\mathbb{R}^2$ and $\mathbb{R}^3$ in certain ways. The functions are called linear transformations, and the domains and ranges are called vector spaces. These are studied in detail in Chaps. 3, 4, and 5. In this section we present a “preview of coming attractions” in the concrete setting of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Let a function $f$ have domain either $\mathbb{R}^2$ or $\mathbb{R}^3$; this function is linear if, for every $x$ and $y$ in the domain of $f$ and every real number $r$,

$$f(x + y) = f(x) + f(y)$$

Tabular 2.3.1 CALCULUS
2.3. MATRICES AS TRANSFORMERS OF SPACE

<table>
<thead>
<tr>
<th>FUNCTION NOTATION</th>
<th>DOMAIN</th>
<th>RANGE</th>
<th>ANALYSIS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = f(x) )</td>
<td>( \mathbb{R} )</td>
<td>( \mathbb{R} )</td>
<td>Derivative, tangent lines max-min, curve sketching, integrals, area between curves</td>
</tr>
<tr>
<td>( z = f(x, y) ) or ( w = f(x, y, z) )</td>
<td>( \mathbb{R}^2 )</td>
<td>( \mathbb{R} )</td>
<td>Partial derivatives, tangent planes, max-min, double or triple integrals, volumes</td>
</tr>
</tbody>
</table>

\[ \mathbf{r}(t) = (f_1(t), f_2(t)) \] or \[ \mathbf{r}(t) = (f_1(t), f(t), f_3(t)) \]
\( \mathbb{R} \) \( \mathbb{R}^2 \) Analysis of curves, vectors, curvature, line integrals
\( \mathbb{R} \) \( \mathbb{R}^3 \)

FIGURE

and

\[ f(rx) = rf(x) \]

Our first example of a linear function comes from the field of mechanics.

**Example 1.** Image a side view of a cube of gelatin held between hands (see Fig. 2.3.1a). If the lower hand is held fixed and the upper hand is moved as in Fig. 2.3.1b, the cube deforms in a special way which leads to what is called **shear**. As an approximation, we assume in Fig. 2.3.1 that the height remains unchanged; this is reasonable if \( k \) is small. We notice in Fig. 2.3.2 that shear transforms \( \mathbf{i} \) to \( \mathbf{i} \) and \( \mathbf{j} \) to \( k\mathbf{i} + \mathbf{j} \) (we have placed the origin at the lower left corner of the face of the cube). In terms of standard coordinate matrices, shear transforms

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} k \\ 1 \end{pmatrix}
\]

The shear function \( \mathcal{S} \) can be represented in standard coordinate matrices

\[
\begin{pmatrix} a \\ b \end{pmatrix}
\]

by

\[
\mathcal{S} \left( \begin{pmatrix} a \\ b \end{pmatrix} \right) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}
\]
We note that under this definition
\[ S\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]
and
\[ S\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k \\ 1 \end{pmatrix} \]
which matches the action shown in Fig. 2.3.2. As defined, the shear function is linear because by the laws of matrix algebra,
\[
S\left(\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}\right) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}\right) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \\
= S\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) + S\left(\begin{pmatrix} c \\ d \end{pmatrix}\right)
\]
and
\[
S\left(r \begin{pmatrix} a \\ b \end{pmatrix}\right) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \left(r \begin{pmatrix} a \\ b \end{pmatrix}\right) = r\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = rS\left(\begin{pmatrix} a \\ b \end{pmatrix}\right)
\]

Shear as a linear function representable by a matrix is only one of several important applications. So it is important to study functions representable by matrices which multiply the standard coordinate matrices of vectors. We discuss several examples and to save words refer to standard coordinate matrices as just vectors.\(^2\) In each example, since the function is defined by matrix multiplication, the linearity follows from the matrix algebra results:
\[
A(X + Y) = AX + AY \\
A(rX) = rAX
\]

**Example 2.** (Production, engineering technology) Suppose a manufacturer makes two kinds of mattresses: soft and firm. The labor per mattress splits according to this chart:

\(^2\)This is a slight abuse of terminology. A three-vector is $1 \times 3$, and its standard coordinate matrix is $3 \times 1$. However, it is easy to make the connection between
\[
\begin{pmatrix} a \\ b \\ c \end{pmatrix}
\]
and is vector $(a, b, c)$.\(^2\)
If \( x_1 \) soft mattresses and \( x_2 \) firm mattresses are produced, the hours of labor used for springs (\( S \)) and padding (\( P \)) are

\[
S = 2x_1 + 3x_2 \\
P = 1x_1 + 2x_2
\]

These equations allow production planning when the amount of labor available for springs and padding is known. We can write this as

\[
\begin{pmatrix}
2 & 1 \\
3 & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
S \\
P
\end{pmatrix}
\]

In this last form, the “labor matrix”

\[
\begin{pmatrix}
2 & 1 \\
3 & 2
\end{pmatrix}
\]

represents a function which acts on a “production vector”

\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

and yields a “labor vector”

\[
\begin{pmatrix}
S \\
P
\end{pmatrix}
\]

**Example 3.** Consider the linear function

\[
f \left( \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} \right) = \begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \begin{pmatrix}
2x_1 \\
2x_2
\end{pmatrix} = 2 \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

For this function, the action of \( f \) on

\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]
is expressed by multiplication of
\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{by the matrix} \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}
\]
We would like to be able to describe what \( f \) does to
\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]
in geometric terms. This is, in fact, the main problem for this section.

**Geometric analysis problem**

Given a linear function \( f \)
\[
f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]
describe geometrically how \( f \) transforms vectors.

**Example 4.** Solve the geometric analysis problem (GAP) for the function from Example 3:
\[
f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

**Solution.** In this case the problem is not as difficult as it will be later. In this case, by direct calculation
\[
f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = 2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]
So the action of \( f \) on a vector is to stretch the vector to twice its original length and leave the direction unchanged. (See Fig. 2.3.3.)

In Example 4 there is nothing special about the constant 2. If \( f \) is defined by
\[
f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]
where \( k > 0 \), then
\[
f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = k \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]
and the action of \( f \) is to stretch \((k > 1)\), shrink \((0 < k < 1)\), or leave unchanged \((k = 1)\) the original vector and, in any case, to leave the direction unchanged. Because of this we introduce this terminology:
If $f$ is defined by a scalar matrix 

\[
\begin{pmatrix}
  k & 0 \\
  0 & k \\
\end{pmatrix}
\]

then it is called

- A **dilation**, if $k > 1$
- The **identity function**, if $k = 1$
- A **contraction**, if $0 < k < 1$
- The **zero function**, if $k = 0$

Now we consider a slightly more complicated function.

**Example 5.** Solve the GAP for the function $P$ defined by

\[
P\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

**Solution** As before, we calculate

\[
P\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}
\]

The effect of $P$ is to wipe out the second component and preserve the first component. To see what this means geometrically, we consider some vectors in Fig. 2.3.4. We see that

\[
P\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)
\]

is the **projection** of

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

onto the $x$ axis. Therefore,
CHAPTER 2. PLANE AND SPACE VECTORS

If
\[
f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]
then \( f \) is a projection onto the \( x \) axis. Similarly if
\[
f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]
then \( f \) is a projection onto the \( y \) axis.

Now we can analyze a slightly more complicated function. Consider
\[
f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]
and calculate
\[
f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 2x_1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \quad \text{Projection}
\]
In this case the original vector is first projected onto the \( x \) axis and then stretched to twice its length. That is, this function is geometrically a projection onto the \( x \) axis, followed by a dilation with constant 2. But the action could also be thought of as the dilation first and projection second.

In terms of the matrices defining the functions, this means that
\[
\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]
\[
\uparrow \\
\text{Dilation} \\
\downarrow \\
\text{Projection}
\]
2nd 1st

but since we also have
\[
\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}
\]
we can write
\[
\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]
\[
\uparrow \\
\text{Projection} \\
\downarrow \\
\text{Dilation}
\]
2nd 1st

This last example illustrates the following general principle:
To solve the geometric analysis problem, break the function $f$ down into a sequence of simple functions such as dilations, contractions, projections, and others.

To use the principle, another simple linear function, rotation, is described.

**Example 6.** For $f$ defined by

$$f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \bullet$$

calculate

$$f \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad f \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

**Solution** We calculate

$$f \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and graph as in Fig. 2.3.5a. Also

$$f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

The graph is shown in Fig. 2.3.5b.

In Example 6 in each case the function gave a vector of the same length and rotated $\pi/2$ radians counterclockwise from the original vector. Of course, this does not prove that the function is a rotation of $\pi/2$ for every vector. However, we can show that the length of a vector is unchanged by this function and that

$$\text{FIGURE}$$

$$f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \perp \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Since

$$f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$
we have
\[ f \left( \frac{x_1}{x_2} \right) = \left| \frac{-x_2}{x_1} \right| = \sqrt{(-x_2)^2 + x_1^2} = \sqrt{x_2^2 + x_1^2} = \left| \frac{x_1}{x_2} \right| \]
and
\[ f \left( \frac{x_1}{x_2} \right) \cdot \left( \frac{x_1}{x_2} \right) = \left( \frac{-x_2}{x_1} \right) \cdot \left( \frac{x_1}{x_2} \right) = -x_2x_1 + x_1x_2 = 0 \]
The last equation says only that
\[ f \left( \frac{x_1}{x_2} \right) \text{ is perpendicular to } \left( \frac{x_1}{x_2} \right) \]
The equation does not tell us that the right angle resulted from rotation. However, it is shown in trigonometry that a counterclockwise rotation of \( x, y \) coordinates through an angle of \( \theta \) is defined by
\begin{align*}
x' &= (\cos \theta)x - (\sin \theta)y \\
y' &= (\sin \theta)x + (\cos \theta)y
\end{align*}
which we can write in matrix form as
\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]
If \( \theta = \pi/2 \), the matrix is
\[ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]
which is exactly the matrix that defines the function we have been working with.

Therefore, the result from trigonometry tells us the following:

A function of the form
\[ f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]
is a rotation counterclockwise through an angle of \( \theta \).

**Example 7.** Identify the counterclockwise angle of rotation for the following functions.
2.3. MATRICES AS TRANSFORMERS OF SPACE

(a) \[ f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

(b) \[ f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

(c) \[ f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

(d) \[ f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

Solution In general, we know that a rotation matrix appears as

\[ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]

To determine \( \theta \), we set \( a = \cos \theta \) and \( b = \sin \theta \) and solve for \( \theta \): \( \theta = \cos^{-1} a \), \( \theta = \sin^{-1} b \). This can be solved by using a table, a calculator, or (in simple cases) memory.

(a) We have \( \cos \theta = 1 \) and \( \sin \theta = 0 \), so \( \theta = 0 \). This makes sense because the function is the identity function.

(b) We have \( \cos \theta = -1 \) and \( \sin \theta = 0 \), so \( \theta = \pi \). This makes sense because

\[ f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} \]

and

\[ \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} \]

is just the reversal of \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \).

(c) In this case \( \cos \theta = \frac{1}{\sqrt{2}} \), and we find \( \theta \) is \( \pi/4 \) or \( 7\pi/4 \). Then using \( \sin \theta = \frac{1}{\sqrt{2}} \), we find that \( \theta \) is \( \pi/4 \) or \( 3\pi/4 \). Since \( \pi/4 \) is the common solution, that is the angle of rotation.
(d) Working as in (c), we have $\cos \theta = -1/\sqrt{2}$ so $\theta$ is $3\pi/4$ or $5\pi/4$. Also $\sin \theta = 1/\sqrt{2}$ which implies that $\theta$ is $\pi/4$ or $3\pi/4$. Therefore the angle of rotation is $3\pi/4$ (counterclockwise).

Now we can solve the geometric analysis problem for some more complicated functions. Consider

$$f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2x_1 \end{pmatrix}$$

This is an interesting function since $f$ composed with itself is

$$f \left( f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) \right) = f \left( \begin{pmatrix} 0 \\ 2x_1 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

That is, if $f$ is applied twice (or more), we get only the zero vector (note that the matrix is nilpotent). To analyze this function, we notice that $x_1$, the first component of

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

is moved by $f$ to the second component. Since this could be done by a rotation of $\pi/2$, we might expect this rotation to the involved. Also since $x_1$ is doubled by $f$, we might expect a dilation with constant 2 also to be involved. We further notice that $f$ destroys $x_2$, so $f$ probably contains a projection onto the $x_1$ axis. Now try to put this information together by representing the individual functions with matrices:

- **Projection onto $x$ axis**: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
- **Dilation with constant 2**: $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
- **Rotation of $\frac{\pi}{2}$**: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

There are several different orders in which these could be applied to

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
For example,
\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

which gives
\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\begin{pmatrix}
-x_2 \\
x_1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
-2x_2 \\
2x_1
\end{pmatrix}
= \begin{pmatrix}
-2x_2 \\
0
\end{pmatrix}
\neq \begin{pmatrix}
0 \\
2x_1
\end{pmatrix} = f\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

So this order is not correct. However,
\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
2x_1 \\
0
\end{pmatrix}
= \begin{pmatrix}
0 \\
2x_1
\end{pmatrix} = f\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

This means that \( f \) consists of first, a dilation with constant 2; second, a projection onto the \( x \) axis; and third, a rotation of \( \pi/2 \). See Fig. 2.3.6

Note that the matrix representing \( f \) satisfies
\[
\begin{pmatrix}
0 & 0 \\
2 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
= \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\]

So the dilation and projection can occur in reverse order. Since matrix multiplication is not commutative, only certain orders of the geometric operations will result in the action of \( f \). In this last example, there are six possible ways to order the operations:

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
2 & 0
\end{pmatrix} \quad \text{Yes}
\]

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
2 & 0
\end{pmatrix} \quad \text{Yes}
\]
We see from this chart that the rotation must occur last.

**Example 8.** Analyze the linear function

\[
f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 3x_2 \end{pmatrix}
\]

**Solution** Note that

\[
\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}
\]

so that

\[
f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

Since \( f \) can be split into the sum of the linear functions \( g \) and \( h \), we know that \( f \) can be analyzed by analyzing \( g \) and \( h \). We have already seen that \( g \) is a projection onto the \( x \) axis followed by a dilation with constant 2. The function \( h \) is similar: It is a projection onto the \( y \) axis followed by a dilation with constant 3. Therefore the action of \( f \) is the vector sum of (1) projection onto \( x \) followed by a dilation, constant 2, and (2) projection onto \( y \) followed by dilation, constant 3. The action of \( f \) is shown in Fig. 2.3.7.

**Example 9.** (Computer graphics) A flat object can be projected on a computer terminal screen because the screen can be coordinatized. That is, a point on the screen can be set as the origin, and \( x \) and \( y \) axes can be declared and scaled. It may be desired to dilate, contract, or rotate the object...
on the screen. This can be done by multiplying the vectors representing the
points on the object by the matrix representing the action, storing the new
vectors, and finally projecting the new points (which are, of course, the ter-
minal points of the vectors). One action which cannot be represented solely
by multiplication by a matrix is translation:

\[
T_{h,k}(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} x_1 + h \\ x_2 + k \end{pmatrix}
\]

FIGURE

<table>
<thead>
<tr>
<th>MATRICES</th>
<th>GEOMETRIC ACTION</th>
</tr>
</thead>
</table>
| \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] | Projection onto x axis, y axis, z axis, respectively |
| \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] | Projection onto xy plane, xz plane, yz plane, respectively |
| \[
\begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\] | Rotation counterclockwise with z fixed |
| \[
\begin{pmatrix}
k & 0 & 0 \\
k & 0 & 0 \\
k & 0 & 0
\end{pmatrix}
\] | \( k > 1 \) Dilation |
| \[
\begin{pmatrix}
k & 0 & 0 \\
k & 0 & 0 \\
k & 0 & 0
\end{pmatrix}
\] | \( k = 1 \) Identity |
| \[
\begin{pmatrix}
k & 0 & 0 \\
k & 0 & 0 \\
k & 0 & 0
\end{pmatrix}
\] | \( 0 < k < 1 \) Contraction |
| \[
\begin{pmatrix}
k & 0 & 0 \\
k & 0 & 0 \\
k & 0 & 0
\end{pmatrix}
\] | \( k = 0 \) Zero function |

Translation is **nonlinear**, because

\[
T_{h,k}(r\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = T_{h,k}(\begin{pmatrix} rx_1 \\ rx_2 \end{pmatrix}) = \begin{pmatrix} rx_1 + h \\ rx_2 + k \end{pmatrix}
\]

and

\[
rT_{h,k}(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} rx + rh \\ rx + rk \end{pmatrix}
\]
The examples so far have involved only linear functions with domain \( \mathbb{R}^2 \). Some simple examples of linear functions with domain \( \mathbb{R}^3 \) are given in Table 2.3.2.

In order to analyze more complicated linear functions generated by matrices or generated by matrices of larger size we need more sophisticated techniques which we will develop later in the text. This will be done by solving the **eigenvalue problem** in Chapter 5.

**Problems 2.3**

1. Solve the geometric analysis problem for the following functions.

   (a) \[ f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

   (b) \[ f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

   (c) \[ f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

2. Solve the geometric analysis problem for the following functions.

   (a) \[ f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

   (b) \[ f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

3. Solve the geometric analysis problem for the following functions.

   (b) \[ f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

   (b) \[ f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

4. Solve the geometric analysis problem for the following functions.

   (a) \[ f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

   (b) \[ f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

5. For linear functions of the form

   \[ f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

   where \( A^{-1} \) exists, we say \( f \) is **invertible** and that

   \[ g \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = A^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]
is the inverse function of \( f \). Show that (a) the inverse of dilation is contraction, (b) the inverse of rotation of angle \( \theta \) is rotation of angle \(-\theta\), and (c) projection onto the \( x \) axis is not invertible.

6. Show geometrically that

\[
f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

represents a reflection about the \( y \) axis. What would

\[
f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

represent?

7. Analyze?

\[
f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} \frac{|x_1|}{|x_2|} \\ x_2 \end{pmatrix}
\]

even though it is not a linear function. (Hint: Consider cases such as \( \{x_1 > 0, x_2 > 0\} \), \( \{x_1 < 0, x_2 > 0\} \), and so on. Is \( f \) “piecewise linear”?)

8. Analyze

\[
f \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} \sqrt{x_1^2 + x_2^2} \\ 0 \end{pmatrix}
\]

even though it is not a linear function. (Hint: Consider that \( f \) does to circles of different radius.)

\[2.4 \quad \textbf{APPLICATIONS TO ANALYTIC GEOMETRY}\]

We can use vectors to derive equations of lines and planes in three-space. Also vectors make the solution of some geometric problems fairly easy.

If we think of a desk top as representing a plane and stand a pencil (unsharpened) on it (see Fig. 2.4.1), the pencil points in a direction perpendicular to any line on the desk top. Such a vector is called a \textbf{normal vector}.

\textbf{Definition 2.4.1.} A vector \( \mathbf{N} \) which is perpendicular to all vectors in a plane \( P \) is called a \textbf{normal vector} for the plane. A normal vector for a plane is said to be perpendicular to the plane.
A plane can be specified by giving a normal to the plane and a point in
the plane. This is similar to the case for lines in the plane for which two
quantities specify a line: the slope and a point. In three-space, the normal
vector serves the function of the slope in two-space. Note that two planes
are parallel if their normals are parallel.

To find an equation for a plane $P$, let the point in $P$ be $(x_0, y_0, z_0)$ and the
normal be $\mathbf{N} = (a, b, c)$. If $(x, y, z)$ is any other point in $P$ (see Fig. 2.4.2),
then the vector $(x - x_0, y - y_0, z - z_0)$ lies in $P$ and $\mathbf{N} \cdot (x - x_0, y - y_0, z - z_0) = 0$
since $\mathbf{N}$ is perpendicular to all vectors in $P$. Writing the dot product, we have

$$0 = \mathbf{N} \cdot (x - x_0, y - y_0, z - z_0) = (a, b, c) \cdot (x - x_0, y - y_0, z - z_0)$$
$$= a(x - x_0) + b(y - y_0) + c(z - z_0)$$

**FIGURE**

Therefore

An equation of the plane containing $(x_0, y_0, z_0)$ with normal $(a, b, c)$
is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This form of the equation is called the **point-normal form** since
the coordinates of the point and components of the normal appear
explicitly. The equation

$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

is called the **vector form** of equation for the plane.

**Example 1.** Find the point-normal form of equation for the plane passing
through $(1, -2, 4)$, having normal vector $(2, 3, -1)$.

**Solution** The vector form is

$$(2, 3, -1) \cdot (x - 1, y - (-2), z - 4) = 0$$

Writing out the dot product, we find

$$2(x - 1) + 3(y + 2) - 1(z - 4) = 0$$
Example 2. A plane $P$ has equation

$$2(x - 2) - 7(x - 3) + 4(x + 2) = 0$$

Find a normal vector to $P$ and a point in $P$.

Solution  Since the equation is in point-normal form, the desired information can be read off:

$$\mathbf{N} = (2, -7, 4)$$

Point (2, 3, -2)

The equation found in Example 1 could be simplified to

$$2x + 3y - z + 8 = 0$$

This equation fits the general form

$$ax + by + cz + d = 0$$

From the general form the normal vector can still be read. Because the equation is linear, a plane is called a linear structure in three-space.

Example 3. A plane $P$ has equation

$$2x - 3y + 4z + 12 = 0$$

Find a normal vector and two points in the plane.

Solution  A normal is (2, -3, 4). Any nonzero multiple of this vector is also a normal. To find points in the plane, we must find values of $x, y,$ and $z$ which satisfy the equation. To do this, we can give values to any two of the variables and solve for the third. If we let $x = 2$ and $y = 0$, we find $z = -4$, so (2, 0, -4) is in the plane. If we let $y = 0$ and $z = 0$, we find $x = -6$ and obtain (-6, 0, 0) as another point in the plane.

The cross product can sometimes be used to find a normal.

Example 4. A plane $P$ contains the points $p_1: (1, 1, 2), p_2: (0, -3, 4)$ and $p_3: (2, 0, 3)$. Find a normal vector for $P$ and an equation for $P$. 
CHAPTER 2. PLANE AND SPACE VECTORS

Solution By subtracting points we find as vectors in \( P: (0, -3, 4) - (1, 1, 2) = \mathbf{A}\) and \( (2, 0, 3) - (1, 1, 2) = (1, -1, 1) = \mathbf{B}\). Now \( \mathbf{A} \times \mathbf{B}\) is perpendicular to both \( \mathbf{A}\) and \( \mathbf{B}\) and hence is normal to \( P\) (see Fig. 2.4.3). Now

\[
\mathbf{A} \times \mathbf{B} = (-2, 3, 5)
\]

A vector equation is

\[
(-2, 3, 5) \cdot (x - 1, y - 1, z - 2) = 0
\]

which is equivalent to

\[-2(x - 1) + 3(y - 1) + 5(z - 2) = 2\]

or

\[-2x + 3y + 5z - 11 = 0\]

A straight line through the point \((x_0, y_0, z_0)\) is the set of all points \((x, y, z)\) such that the vector from \((x_0, y_0, z_0)\) to \((x, y, z)\) is a multiple of a given vector

\[
\text{FIGURE}
\]

\(\mathbf{V}\) (see Fig. 2.4.4). From Fig. 2.4.4 we see that

\[
(x - x_0, y - y_0, z - z_0) = k\mathbf{V} = k(a, b, c)
\]

where \(k\) is allowed to run through the real numbers, which means

\[
x - x_0 = ka \quad y - y_0 = kb \quad z - z_0 = kc
\]

(parametric equations)

or

\[
\frac{x - x_0}{a} = k = \frac{y - y_0}{b} = k = \frac{z - z_0}{c}
\]

(unless \(a, b,\) or \(c\) is zero).

Example 5. Find equations of the line passing through \((1, -1, 2)\) and \((7, 0, 5)\). Is \((-5, -2, -1)\) on the line? What about \((13, 1, 4)\)?

Solution We have a point (in fact, we have two) and need a vector in the direction of the line. For this purpose a vector in the line will do. So we subtract the points to get

\[
\mathbf{V} = (6, 1, 3)
\]
Using \((1, -1, 2)\) as a point in the line, we see that
\[
(x - 1, y + 1, z - 2) = k(6, 1, 3)
\]
is a vector equation for the line. Parametric equations are
\[
\begin{align*}
x - 1 &= 6k \\
y + 1 &= k \\
z - 2 &= 3k
\end{align*}
\]
or
\[
\begin{align*}
x &= 6k + 1 \\
y &= k - 1 \\
z &= 3k + 2
\end{align*}
\]
To see whether \((-5, -2, -1)\) is on the line, we ask if there is a value for \(k\) (the parameter) which gives \(x = -5, y = -2, z = -1\). Using the last equations, we try to solve
\[
\begin{align*}
-5 &= 6k + 1 \\
-2 &= k - 1 \\
-1 &= 3k + 2
\end{align*}
\]
The first equation gives \(k = -1\). This value for \(k\) also works in the second and third equations so \((-5, -2, -1)\) is on the line.

Finally we consider the point \((13,1,14)\). We have this time
\[
\begin{align*}
13 &= 6k + 1 \\
1 &= k - 1 \\
14 &= 3k + 2
\end{align*}
\]
and the first equation implies that \(k = 2\). This value for \(k\) satisfies the second equation but not the third. Thus \((13,1,14)\) is not on the line.

Lines can be generated as the intersection of planes. In general, two planes can intersect in one line, not intersect at all, or be the same plane.

**Example 6.** Determine the line of intersection of \(x + 2y - z = 3\) and \(x + y - 3z = 5\).

**Solution** We solve the equations simultaneously:
\[
\begin{align*}
x + 2y - z &= 3 \\
x + y - 3z &= 5 \\longrightarrow \quad x + 2y - z &= 3 \\
y - 2z &= 2
\end{align*}
\]
We have two equations in three unknowns. Let \( z = t \), so that
\[
\begin{align*}
y &= -2t - 2 \\
x &= 5t + 7
\end{align*}
\]
Thus, parametric equations for the line of intersection are
\[
x = 5t + 7 \quad y = -2t - 2 \quad z = t \quad t \in \mathbb{R}
\]
**Example 7.** Show that the planes \( x - y + z = 3 \) and \( 3x - 3y + 3z = 11 \) do not intersect.

**Solution** Again we consider the equations simultaneously:
\[
\begin{align*}
x - y + z &= 3 \\
3x - 3y + 3z &= 11 \quad \rightarrow \quad x - y + z &= 3 \\
0 &= 2
\end{align*}
\]
Since the reduced system is inconsistent, no points lie on both planes (they are parallel, since their normals are in the same direction) and so there is no line of intersection for these planes.

Note in Example 7 that if the second plane had had equation
\[
3x - 3y + 3z = 9
\]
then the planes would be not only parallel but also coincident, and there would be no unique line of intersection.

The use of vectors in the study of analytic geometry is a very effective method. Interested readers can find further information in texts on **vector geometry**.

**Problems 2.4**

1. Find vector and general forms of equations for the planes given. Sketch the planes.
   - (a) Point: \((1, -2, 1)\), normal: \((1, 2, 3)\)
   - (b) Point: \((0, 0, 0)\), normal: \((3, -1, 0)\)
   - (c) Point: \((-2, -3, 4)\), normal: \((1, 0, 0)\)
   - (d) Point: \((1, 0, 0)\), normal: \((-1, -1, -1)\)

2. For the following planes state a normal vector to the plane, a unit normal to the plane, and two points in the plane. Sketch the plane.
2.4. APPLICATIONS TO ANALYTIC GEOMETRY

(a) \( x - 3y + z = 7 \)  
(b) \( 3x - 4y - 6z = 18 \)  
(c) \( -5x + 6y + z = 10 \)  
(d) \( x = 2 \)

3. Three points are given. Find vector forms and general forms of equation for the planes passing through the points. Sketch the planes.

(a) \( (1, -1, 0) \)  \( (2,1,3) \)  \( (4,6,5) \)  
(b) \( (-1, 0, 0) \)  \( (0, -1, 0) \)  \( (0, 0, -1) \)  
(c) \( (2,4,6) \)  \( (2, -1, 3) \)  \( (2,7,13) \)

4. Write equations for the line parallel to \( \mathbf{V} \) and passing through the point \( P \). Sketch the line.

(a) \( \mathbf{V} = (1, -1, 3), P: (0, 4, 7) \)  
(b) \( \mathbf{V} = (1,1,0), P: (-7, 1, 6) \)  
(c) \( \mathbf{V} = (-1, -1, -2), P: (0,0,0) \)

5. Write equations for the lines passing through the given points. Sketch the lines.

(a) \( (1,0,0) \)  \( (0,1,0) \)  
(b) \( (0,0,0) \)  \( (1,1,1) \)  
(c) \( (1,1,0) \)  \( (0,0,1) \)  
(d) \( (1,0,1) \)  \( (0,1,1) \)

6. Let \( l \) be the line passing through \( (1, -1, 4) \) and \( (2, 4, -2) \). Which of the following points lie on \( l \)?

(a) \( \left(\frac{3}{2}, \frac{3}{2}, 1\right) \)  
(b) \( (1,5, -6) \)  
(c) \( (3,5,7) \)  
(d) \( (0, -6, 11) \)  
(e) \( (0, -6, 10) \)

7. Write equations for the (special) planes: the \( xy \), \( xz \), and \( yz \) planes.

8. Determine whether the following pairs of planes intersect. If they do and are not the same plane, give an equation for the line of intersection. Sketch the planes.

(a) \( x - 3y + 4z = 11; 2x + 7y - z = 7 \)  
(b) \( x - 2y + z = 7; -5x + 10y - 5z = -30 \)
(c) $x = 2, x + y + z = 2$

9. Show that the line

\[
\begin{align*}
x &= 2t - 6 \\
y &= t + 4 \quad t \in \mathbb{R} \\
z &= 3t - 6
\end{align*}
\]

is parallel to

\[
3x - 3y - z = 6
\]

by showing (a) that the normal to the plane is perpendicular to the line and (b) that points of form (1) do not satisfy Eq. (2) for any $t$.

10. Show that the line

\[
\begin{align*}
x &= t - 3 \\
y &= 2t + 1 \\
z &= t + 6
\end{align*}
\]

intersects the plane

\[
x - 2y + z = 6
\]

by substituting the expressions from (3) into (4) and solving for $t$. State the point of intersection. Sketch the line and plane.

**SUMMARY**

We have reviewed and reinforced the reader’s knowledge of elementary vector analysis; also we have introduced the idea of a linear function generated by a matrix. The sum and difference of vectors were given analytical definitions and geometric meaning. The dot product and corresponding idea of perpendicularity were developed.

In space and the plane, we have geometry to reinforce our analytic results about lines, planes, and linear functions. The first paragraph of this chapter points out that we must leave this comfortable situation now and make the leap into higher dimensions. Although we lose some geometric insight in our new setting and gain corresponding problems, the applications are rich and meaningful.
The point of departure in Chap. 3 lies in the generalization of our ideas of two and three vectors. This will give us new domains for linear functions of applied mathematics.

The idea of the geometric analysis of a linear function was introduced, and several examples were given. The idea of studying function geometrically is not new to the reader; after all, in high school algebra one learns to graph functions with domain and range in the real numbers. A graph is simply a geometric representation of a function. In this chapter we did not “graph” the linear functions in the examples. Instead we described their action in terms of definite geometric operations. As long as we stay in the plane or space, the ideas of projection, rotation, dilation, contraction, and reflection have a definite geometric, visual meaning for us. When we move to the higher dimensions mentioned above, we may still use the words projection, dilation, and so on, even though we cannot “see” what is happening (in five dimensions, for instance). Because of this it is important for us to now set up the machinery to study extended notions of projections, dilations, and so forth.

ADDITIONAL PROBLEMS

1. Consider the diagram of planar forces pulling on the origin in Fig. AP2.1. The forces are in equilibrium; that is, $\mathbf{A} + \mathbf{B} + \mathbf{C} = 0$. What is the vector $\mathbf{C}$?

2. If two planar forces are acting on a point in the plane, can one always impose a third force to create equilibrium? That is, can one always make the resultant the zero vector?

3. The principle of a teeter-totter is, from the picture in Fig. AP2.2, that the teeter-totter will balance if $w_1 x = w_2 y$. The principle also applies to the hanging weights in Fig. AP2.3: they will balance if $w_1 x = w_2 y$. For the more complicated system of hanging weights in Fig. AP2.4, the balancing equations, by the same principle, are

$$7w_1 = 6w_2 + 11w_3$$
$$2w_2 = 3w_3$$
$$7w_1 = 8(w_2 + w_3)$$
Rewrite the equations as a homogeneous system. Solve the system. What can you say about the choice of weights to make the physical system balance?

4. Regarding the hanging weights in Prob. 3, suppose you were told that $w_1 = 5$ lb. What would $w_2$ and $w_3$ be?

5. For the system of weights in Fig. AP2.5, the equations of balance are

\[
\begin{align*}
  w_3 &= w_1 + w_2 \\
  2w_1 &= kw_2 \\
  5w_3 &= 7w_1 + (5 - k)w_2
\end{align*}
\]

What value of $k$ guarantees a nontrivial solution and therefore a set of weights to balance the system?

6. Show that if each component planar force acting on a point is doubled, then the resultant is doubled. Use a matrix to represent the dilation.

7. Show that if one rotates by $\theta$ each component planar force acting on a point, then the resultant is rotated by that same angle.

8. Show that if you reflect about the $x$ axis each component planar force acting on a point, then the resultant is reflected about the $x$ axis.

9. Show that if you contract (multiply by $k$, $0 < k < 1$) each component planar force acting on a point, then the resultant force is similarly contracted.

10. We have discussed matrices as transformers of space. Matrices also act as transformers of outputs to inputs, as we saw in Chap. 1. The matrix

\[
A = \begin{pmatrix}
1 & R \\
0 & 1
\end{pmatrix}
\]

represents the box in Fig. AP2.10. Thus we have

\[
\begin{pmatrix}
V_1 \\
I_1
\end{pmatrix} = \begin{pmatrix}
1 & R \\
0 & 1
\end{pmatrix} \begin{pmatrix}
V_2 \\
I_2
\end{pmatrix}
\]
If we calculate $A^{-1}$, we have a matrix which transforms the inputs to the outputs:

$$A^{-1} \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = (A^{-1}A) \begin{pmatrix} V_2 \\ I_2 \end{pmatrix} = \begin{pmatrix} V_2 \\ I_2 \end{pmatrix}$$

Calculate $A^{-1}$. Note that $A^{-1}$ has some negative entries and is in the same upper triangular form as $A$. Whether we can construct a blackbox represented by $A^{-1}$ depends on what we can put in the box.