Math 640 — Homework 3

1. Which of the following are subspaces of the set of all \( n \times n \) matrices over the reals, which we denote by \( M_n(\mathbb{R}) \).
   a. \( V = \{ A \in M_n(\mathbb{R}) \mid A \text{ is upper triangular} \} \)
   b. \( V = \{ A \in M_n(\mathbb{R}) \mid A \geq 0 \text{ (all entries are nonnegative)} \} \)
   c. \( V = \{ A \in M_n(\mathbb{R}) \mid a_{ij} = 0 \text{ if both } i \text{ and } j \text{ are odd integers} \} \)
   d. \( V = \{ A \in M_n(\mathbb{R}) \mid A^m = 0 \text{ for some positive integer } m \} \) (This one, about nilpotent matrices, is a little tricky.)

2. We say a matrix \( A \in M_n(\mathbb{R}) \) is orthogonal if \( A^T = A^{-1} \).
   a. Show that orthogonal matrices have orthogonal rows and orthogonal columns.
   b. Show that if \( A \) is orthogonal then \( A^m \) is orthogonal for every positive integer \( m \).

3. Let \( V \) be a vector space with an inner product \( \langle , \rangle \) and \( \{ y_1, \ldots, y_k \} \subseteq V \). Let \( x \perp y \) mean that \( \langle x, y \rangle = 0 \). Show that if \( x \perp y_1, \ldots, x \perp y_k \), then \( x \) is orthogonal to every vector in the span of \( \{ y_1, \ldots, y_k \} \).

4. Solve the following system \( Ax = b \) by first finding the RREF of the augmented matrix.
   \[
   A = \begin{bmatrix}
   3 & 3 & -2 \\
   1 & -1 & 1 \\
   3 & -2 & -2 \\
   \end{bmatrix}, \quad b = \begin{bmatrix}
   -1 \\
   1 \\
   1 \\
   \end{bmatrix}
   \]

5. Use the Gram-Schmidt process to find an orthogonal set of vectors that spans the subspace of \( \mathbb{R}^3 \) generated by the vectors \( \{ [1,2,1], [-1,2,0], [0,2,0] \} \).

6. Show that the set \( \left\{ \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & -1 \\ 3 & 0 \end{bmatrix} \right\} \) is linearly dependent in \( M_2 \).

7. What are the signs of the permutations \{7, 6, 5, 4, 3, 2, 1\} and \{7, 1, 6, 4, 3, 5, 2\} of the integers \{1, 2, 3, 4, 5, 6, 7\}?

8. Using minors, find the determinant of
   \[
   A = \begin{bmatrix}
   1 & 0 & 1 & 0 \\
   0 & 1 & 1 & -1 \\
   1 & 0 & 1 & 1 \\
   1 & 0 & 0 & -1 \\
   \end{bmatrix}
   \]