1. We know a diagonal matrix has zero entries off the diagonal. An antidiagonal matrix $A$ satisfies $a_{i,j} = 0$ if $n-i+1$. Thus an antidiagonal matrix looks like this:

$$
\begin{bmatrix}
0 & \cdots & 0 & a_{1n} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & a_{2,n-1} \\
0 & \cdots & 0 & a_{n-1,2} \\
0 & \cdots & 0 & a_{n1}
\end{bmatrix}
$$

Prove the following:

(a) Characterize the product of two antidiagonal matrices.

(b) Antidiagonal matrices are invertible if and only if all the antidiagonal components are nonzero. That is, $a_{i,n-i+1} \neq 0$ for $i = 1, \ldots, n$.

(c) Let $\mathcal{D}$ and $\mathfrak{A}$ denote the invertible diagonal and antidiagonal matrices, respectively. Show that $\mathcal{D} \cup \mathfrak{A}$ forms a group.

2. We know that the spectrum of $A \in M_3(\mathbb{C})$ is $\sigma(A) = \{1, 1, -2\}$ and the corresponding eigenvectors are $\{[1,2,1]^T, [2,1,-1]^T, [1,1,2]^T\}$.

(i) Is it possible to determine $A$? Why or why not? If so, prove it. If not, show two different matrices with the given spectrum and eigenvectors.

(ii) Is $A$ diagonalizable?

3. Prove that if $A \in M_n(\mathbb{C})$ is diagonalizable then for each $\lambda \in \sigma(A)$, the algebraic and geometric multiplicities are equal. That is, $m_{a}(\lambda) = m_{g}(\lambda)$.

4. A matrix $A \in M_n(\mathbb{C})$ has a square root if there is a matrix $B \in M_n(\mathbb{C})$ such that $B^2 = A$. Show that if $A$ is diagonalizable then it has a square root.

5. Suppose that if $A \in M_n(\mathbb{C})$ is diagonalizable and for each $\lambda \in \sigma(A)$, $|\lambda| < 1$. Prove directly from similarity ideas that $\lim_{n \to \infty} A^n = 0$.(That is, do not apply the more general theorem from the lecture notes.)

6. We say that $A$ is right quasi-invertible if there exists a matrix $B \in M_n(\mathbb{C})$ such that $AB \sim D$, where $D$ is a diagonal matrix with diagonal entries nonzero. Similarly, we say that $A$ is left quasi-invertible if there exists a matrix $B \in M_n(\mathbb{C})$ such that $BA \sim D$, where $D$ is a diagonal matrix with diagonal entries nonzero.
(i) Show that if $A$ is right quasi-invertible then it is invertible.

(ii) Show that if $A$ is right quasi-invertible then it is left quasi-invertible.

(iii) Prove that quasi-invertibility is not an equivalence relation. (Hint. How do we usually show that an assertion is false?)