Math 640 — Midterm practice examination

1. Give a proof that if $A, B \in M_n(\mathbb{C})$ and $AB$ has rank $n$, then $A$ has rank $n$.

2. We say a matrix $A \in M_n(\mathbb{C})$ has a square root $B \in M_n(\mathbb{C})$ if $B^2 = A$. Prove that every diagonalizable matrix has a square root.

3. We say a matrix $A \in M_n(\mathbb{C})$ is diagonally dominant if
   \[ \sum_{j=1, j\neq i}^{n} |a_{ij}| < |a_{ii}| \text{ for all } i = 1, \ldots, n \]
   (i) Prove diagonally dominant matrices are invertible.
   (ii) Prove that if $A$ is diagonally dominant, then $\rho(A) < 2 \max_{1 \leq i \leq n} |a_{ii}|$.

4. Prove that if $A \in M_{m,p}$ and $B \in M_{p,n}$ then $r(AB) \geq r(A) + r(B) - n$.

5. Suppose $A \in M_n(\mathbb{C})$ Define the function defined on $M_n(\mathbb{C})$ by $\|A\| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$.
   (i) Prove that $\|\cdot\|$ is a norm on $M_n(\mathbb{C})$.
   (ii) Prove that $\|\cdot\|$ is the matrix norm pertaining the $\|\cdot\|_\infty$ norm on $\mathbb{C}^n$.

6. Suppose $A \in M_n(\mathbb{C})$ and there is an invertible matrix $S \in M_n(\mathbb{C})$, for which $S^{-1}AS = D$ is a diagonal matrix. Prove that the columns of $S$ are the eigenvectors of $A$. (Give a complete proof.)

7. We say a matrix $A \in M_n(\mathbb{R})$ is column stochastic if all its elements are nonnegative and
   \[ \sum_{j=1}^{n} a_{ij} = 1 \text{ for all } i = 1, \ldots, n \]
   (i) Prove that the set of column stochastic matrices is multiplicatively closed.
   (ii) Prove that the spectral radius $\rho(A) = 1$.

8. Let $A, D \in M_n(\mathbb{R})$ where $D$ is diagonal with distinct diagonal elements. Prove that $A$ and $D$ commute if and only if $A$ is also diagonal.

9. Let $A \in M_n(\mathbb{R})$ be skew-symmetric. Show that $A$ is diagonalizable.

10. Prove that Unitary matrices are closed in the pointwise topology. That is, if $\{U_n\}$ is a sequence of unitary matrices and $\lim U_n = V$ for each entry, then $V$ is unitary.
11. Which Householder transformations on $C_2$ are rotations?
12. Show that for every matrix in $A \in M_n$ there is a lower triangular matrix $L$ and an upper triangular matrix $U$ such that $LA = U$
13. Prove that the unitary matrices are closed the norm. That is, if the sequence
\[ \{ U_n \} \subset M_n(C) \] are all unitary and $\lim_{n \to \infty} U_n = U$ in the $\| \cdot \|_2$ norm, then $U$ is also unitary.
14. Given two finite sequences $\{ c_k \}_{k=1}^n$ and $\{ d_k \}_{k=1}^n$. Prove that $\sum_k |c_k d_k| \leq \max_k |d_k| \sum_k |c_k|$.
15. Verify that a matrix norm which is subordinate to a vector norm satisfies norm conditions
(i) and (ii).
16. Let $A \in M_n(C)$. Show that the matrix norm subordinate to the vector norm $\| \cdot \|_\infty$ is given by
\[ \| A \|_\infty = \max_i \| r_i(A) \|_1 \]
where as usual $r_i(A)$ denotes the $i^{th}$ row of the matrix $A$ and $\| \cdot \|_1$ is the $\ell_1$ norm.
17. The Hilbert matrix, $H_n$ of order $n$ is defined by
\[ h_{ij} = \frac{1}{i+j-1} \quad 1 \leq i, j \leq n \]
18. Prove or disprove that $\| H_n \|_1 < \ln n$.
19. Show that $\| H_n \|_\infty = 1$.
20. Show that $\| H_n \|_2 \sim n^{\frac{1}{2}}$ as $n \to \infty$.
21. Show that the spectral radius of $H_n$ is bounded by 1.
22. Show that for each $\varepsilon > 0$ there exists an integer $N$ such that if $n > N$ there is a vector $x \in R_n$ with $\| x \|_2 = 1$ such that $\| H_n x \|_2 < \varepsilon$.
23. Same as the previous question except that you need to show that $N = \left\lfloor \frac{1}{\varepsilon^{1/2}} \right\rfloor + 1$ will work.
24. Prove that the product of two isometries is an isometry.
25. Let $A \in M_n(C)$. If $\text{tr}(A^k) = 0$, $k = 1, \ldots, n$, then $A^n$ is nilpotent.
26. Let $A \in M_n(C)$. If $-1 \not\in \sigma(A)$ we can define the Cayley transform of $A$ by
\[ C(A) = (I + A)^{-1}(I - A) \] Show that if $A$ is skew-Hermitian, then $C(A)$ is unitary and conversely.
27. Let $A \in M_n(C)$. Prove that if $\sigma(A)$ is contained in the left half plane, then $C(A)$ is convergent. That is $\lim_{n \to \infty} (C(A))^n = 0$. (This is a little difficult.)