1. Consider the arithmetic progression 0, b, 2b, 3b, . . . . Suppose \((d, b) = 1\).\(^2\) Prove that the series \(\{kb \pmod{d}\}, k = 1, 2, . . . ,\) contains \(d\) different residues. (Hint. Prove that the series \(\{kb \pmod{d}\}, k = 1, 2, . . . , d\) contains \(d\) different residues.

2. For any integers \(a, b, \) and \(m\) show that \(ab \pmod{m} = [(a \pmod{m})b \pmod{m}] \pmod{m}\).

3. Given an argument that when constrained to a fixed mantissa arithmetic, that every mathematical formula proposed to generate random numbers must cycle.

4. The floor function takes any non-integer number to the next smaller integer, while leaving integers unchanged. For example, \(\lfloor 3 \rfloor = 3, \lfloor -3.19 \rfloor = -4, \lfloor 7.939 \rfloor = 7\). Using the floor function, give a formula for the middle-square algorithm.

5. How can you utilize random numbers in the classroom to illustrate some mathematical concept?


7. Let \(s = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{12} + \frac{1}{15} + \frac{1}{16} + \cdots\) be the sum of the reciprocals of all numbers with primes factors 2, 3, and 5. Prove Euler’s formula in the special case, that \(\prod_{p=2,3,5} \frac{1}{1-p}
\)

8. Compute \(\varphi(25), \varphi(32),\) and \(\varphi(100)\).

9. Show that \(\varphi(2n) = \varphi(n)\), for every odd integer \(n\).

10. Prove that if the integer \(n\) has \(r\) distinct primes, the \(2^2 | \varphi(n)\).

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\(^2\)This means \(d\) and \(b\) are relatively prime.
11. Prove that the Euler \( \varphi \)-function is multiplicative. That is, \( \varphi(mn) = \varphi(m) \varphi(n) \). (This may prove difficult.)

12. Show that there is no odd perfect number that is the product of just two odd primes (¿1).

13. Prove the formula \( \ln (1 - x^2) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) \)

14. Express \( \sqrt{i} \) in the form \( a + ib \).

15. Classify which numbers of the form \( \sqrt{p^q} \) are transcendental.

16. Use the classical result \( e^{i\pi} = -1 \) and Gelfond’s theorem to establish that \( e \) cannot be algebraic.

17. Note two example of aspects of number theory that required further algebraic development ot solve.

18. Explain the development of algebra as a consequence of symbolism. (Hint. What aspects of 19th century developments would have been impossible without symbolism?)

19. Write a short essay on the impact of number theory on the development of algebra.