Solutions to Homework II: 7.2-7.3

1. (7.2 #5) Find the volume of the solid obtained by rotating the region bounded by \( y = x^2, \ y = 4, \ x = 0, \) and \( x = 2 \) about the \( y \)-axis.

   The function can be written in terms of \( x \) or \( y \). Since there is no hole, we use slices and write our function in terms of \( y \). Partition \([0, 4]\) along the \( y \)-axis. The volume of a slice is \( \pi r^2 h \), where \( r = x = \sqrt{y} \) and \( h = dy \). Therefore, the integral which represents the volume of the solid is

   \[
   V = \int_0^4 \pi (\sqrt{y})^2 dy \\
   V = \pi \int_0^4 y \ dy \\
   V = \pi \left( \frac{1}{2} \right) y^2\bigg|_0^4 = 8\pi
   \]

2. Find the volume of the solid formed by rotating the region enclosed in the first quadrant by \( \frac{x}{r} + \frac{y}{h} = 1 \) about the \( y \)-axis.

   Again the function can be written in terms of \( y \) or \( x \) and we are rotating about the \( y \) axis. The algebra is slightly easier if we write the function of \( x \) and use shells. Partition \([0, r]\) on the \( x \) axis. The volume of a shell is \( 2\pi rh \Delta r \), where \( r = x, \ \Delta r = dx, \) and \( h = f(x), \) or \( h \left( 1 - \frac{x}{r} \right) \). Therefore, the integral which represents the volume of the solid is

   \[
   V = \int_0^r 2\pi xh \left( 1 - \frac{x}{r} \right) \ dx \\
   V = 2\pi h \int_0^r \left( x - \frac{x^2}{r} \right) \ dx \\
   V = 2\pi h \left( \frac{1}{2} x^2 - \frac{1}{3r} x^3 \right)\bigg|_0^r \\
   V = 2\pi h \left( \frac{1}{2} r^2 - \frac{1}{3r} r^3 \right) \\
   V = 2\pi h \left( \frac{1}{6} r^2 \right) = \frac{1}{3} \pi r^2 h
   \]

   Note that if you use slices, your integral will be

   \[
   V = \int_0^h \pi \left( r \left( 1 - \frac{y}{h} \right) \right)^2 \ dy
   \]

3. (7.3 #13) Find the volume of the solid obtained by rotating the region bounded by \( y = \sqrt{x}, \ y = 0, \) and \( x + y = 2 \) about the \( x \)-axis.

   If we use slices, our radius changes during the interval, so we should use shells. Since we are rotating about the \( x \) axis, our functions should be written in terms of \( y \), so we have \( x = y^2 \) and \( x = 2 - y \). These curves intersect when

   \[
   y^2 = 2 - y \\
   y^2 + y - 2 = 0 \\
   (y + 2)(y - 1) = 0
   \]
or \( y = -2 \) and \( y = 1 \). Since the region is bounded below by \( y = 0 \), we partition the interval \([0, 1]\) on the \( y \) axis. The volume of a shell is given by \( 2\pi rh\Delta r \), where \( r = y, \Delta r = dy \), and \( h = (2 - y) - y^2 \) (Right - Left). Therefore, our integral to find the volume is given by

\[
V = \int_0^1 2\pi y(2 - y - y^2) \, dy
\]

\[
V = 2\pi \int_0^1 (2y - y^2 - y^3) \, dy
\]

\[
V = 2\pi \left( y^2 - \frac{1}{3}y^3 - \frac{1}{4}y^4 \right) \bigg|_0^1
\]

\[
V = 2\pi \left( 1 - \frac{1}{3} - \frac{1}{4} \right) = \frac{5\pi}{6}
\]

4. (7.3 #30) The integral \( \int_0^\pi 2\pi(4 - x) \sin 4x \, dx \) represents the volume of a solid. Describe the solid.

Since we have a \( 2\pi \) in front of the function and a (shifted) \( x \) before the function, we suspect that the integral is set up using the shell method. The format of a shell-integral is

\[
V = \int_a^b 2\pi rh \Delta r
\]

So matching this up with the given integral, we see that \( r = 4 - x, \Delta r = dx \), and \( h = \sin 4x \). Since \( h \) is found by Top - Bottom, the region is bounded above by \( y = \sin 4x \), below by \( y = 0 \), left by \( x = 0 \), and right by \( x = \pi \). Since the radius is \( 4 - x \), this means we are rotating around the vertical line \( x = 4 \) (the entire region is to the left of \( x = 4 \), so that is why we subtract \( 4 - x \). Therefore, the solid is formed by rotating the region bounded by \( y = \sin 4x \), \( y = 0 \), \( x = 0 \), and \( x = \pi \) about the line \( x = 4 \).

5. There are several formulas in these sections for rotating regions about the \( x \) or \( y \) axis. Given \( f(x) \geq 0 \) on \([a, b]\) and \( k < 0 \), use an appropriate Riemann Sum to derive a formula for rotating the region under the graph of \( y = f(x) \) about the line \( y = k \). Explain why this formula will not work when \( k \) is positive.

Since our function is given in terms of \( x \) (since \( f \) is not necessarily one-to-one, we CAN- NOT let \( x = g(y) \)) and our axis is parallel to the \( x \) axis, we can use slices/washers (outer-inner). The volume of a washer is given by \( \pi r_{outer}^2 h - \pi r_{inner}^2 h \), where \( r_{outer} = f(x_i^*) - k \), \( r_{inner} = -k (0 - k) \), and \( h = \Delta x_i \). Therefore, our volume is given by

\[
V = \lim_{|P| \to 0} \sum_{i=1}^n \pi \left( (f(x_i^*) - k)^2 - k^2 \right) \Delta x_i
\]

Which is a Riemann Sum definition for

\[
V = \int_a^b \pi (f(x) - k)^2 \, dx - \int_a^b \pi k^2 \, dx
\]

These can also be combined into a single integral:

\[
V = \pi \int_a^b ((f(x) - k)^2 - k^2) \, dx
\]

There are 2 potential problems with the formula when \( k \) is positive:
First, if \( y = k \) cuts through the region, then there is no hole in the solid (and therefore
no subtraction in the integral) at all. The solid is still defined, but the volume is given by

\[ V = \pi \int_{a}^{b} |f(x) - k|^2 \, dx \]

If \( y = k \) lies above the solid, then \( r_{outer} = k \) and \( r_{inner} = k - f(x) \), so the integrals would be reversed:

\[ V = \pi \int_{a}^{b} (k^2 - (k - f(x))^2) \, dx \]