Solutions to Homework IV: 8.4, 8.8, 8.9

1. (8.4 #25) Evaluate \( \int_{0}^{1} \frac{2x + 3}{(x + 1)^2} \, dx \)

Use partial fraction decomposition:
\[
\frac{2x + 3}{(x + 1)^2} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2}
\]
2x + 3 = A(x + 1) + B

If \( x = -1 \):
1 = B

If \( x = 0 \) (or matching powers), 3 = A + B, or A = 2.

\[
\int_{0}^{1} \frac{2x + 3}{(x + 1)^2} \, dx = \int_{0}^{1} \left( \frac{2}{x + 1} + \frac{1}{(x + 1)^2} \right) \, dx
\]

= \( 2 \ln |x + 1| - \frac{1}{x + 1} \bigg|_{0}^{1} \)

= \( 2 \ln 2 - \frac{1}{2} - 2 \ln 1 + 1 = 2 \ln 2 + \frac{1}{2} \)

2. Find all values of \( p \) for which \( \int_{0}^{1} \frac{1}{x^p} \, dx \) converge

\[
\int_{0}^{1} \frac{1}{x^p} \, dx = \lim_{t \to 0^+} \int_{t}^{1} x^{-p} \, dx
\]

If \( p \neq 1 \),

= \( \lim_{t \to 0^+} \frac{1}{-p+1} x^{-p+1} \bigg|_{t}^{1} \)

= \( \lim_{t \to 0^+} \frac{1}{-p+1} - \frac{1}{-p+1} t^{-p+1} \)

The limit will exist if and only if the “t” term “stays in the numerator”, i.e., when \( -p + 1 > 0 \), or \( p < 1 \). Therefore, the integral converges when \( p < 1 \) and diverges when \( p > 1 \).

When \( p = 1 \), we have

\[
\int_{0}^{1} \frac{1}{x^p} \, dx = \lim_{t \to 0^+} \int_{t}^{1} \frac{1}{x} \, dx
\]

= \( \lim_{t \to 0^+} \ln x \bigg|_{t}^{1} \)

= \( \lim_{t \to 0^+} \ln 1 - \ln t \)

As \( t \) approaches 0, \( \ln t \) approaches \(-\infty\), so the integral diverges.

Therefore, the integral converges if and only if \( p < 1 \).
3. (8.9 #67c) The Laplace transform of a continuous function \( f \) is defined by

\[
F(s) = \int_0^\infty f(t)e^{-st} \, dt
\]

Find the Laplace transform of \( f(t) = t \).

Replacing \( f(t) \) in the definition, we have

\[
F(s) = \int_0^\infty te^{-st} \, dt
\]

Use integration by parts with \( u = t, \ dv = e^{-st} \). Then \( du = dt, \ v = -\frac{1}{s}e^{-st} \). The integral of \( vu \) becomes

\[
-\frac{1}{s} \int e^{-st} \, dt = \frac{1}{s^2}e^{-st}, \text{ so our original integral becomes}
\]

\[
= \lim_{b \to \infty} \left. \frac{-1}{s} e^{-st} \right|_0^b
= \lim_{b \to \infty} \left( -\frac{1}{s} e^{-sb} + \frac{1}{s^2} e^{0b} \right)
\]

As \( b \) approaches \( \infty \), the second term approaches 0, but the first term is not clear (indeterminate: \( \infty \cdot 0 \)), so we use L'Hospital's Rule:

\[
\lim_{b \to \infty} -\frac{1}{s} be^{-sb} = \lim_{b \to \infty} -\frac{1}{s^2} e^{sb} = 0.
\]

Therefore, the integral converges to \( F(s) = \frac{1}{s^2} \).

4. The integral \( \int_0^1 \frac{4}{1 + x^2} \, dx \) is approximated using the midpoint rule with \( n \) rectangles. Determine the largest possible error as a function of \( n \). Explain clearly how you obtained an upper bound on \( |f''(c)| \).

\[
E_M = \frac{f''(c)(b - a)^3}{24n^2}. \text{ Here } a = 0 \text{ and } b = 1, \text{ and we need a bound on } f''(c).
\]

\[
f(x) = \frac{4}{1 + x^2} = 4(1 + x^2)^{-1}
\]

\[
f'(x) = -8(1 + x^2)^{-2}(2x) = -8x(1 + x^2)^{-2}
\]

Using the product rule, \( f''(x) = -8(1 + x^2)^{-2} - 8x(-2)(1 + x^2)^{-3}(2x) \)

\[
f''(x) = \frac{-8}{(1 + x^2)^2} + \frac{32x^2}{(1 + x^2)^3}. \text{ Since we do not know how the function behaves on } [0, 1],
\]

we simplify into a single fraction with a common denominator:

\[
f''(x) = \frac{-8(1 + x^2) + 32x^2}{(1 + x^2)^3} = \frac{24x^2 - 8}{(1 + x^2)^3}.
\]

Since the denominator is smallest when \( x = 0 \), we know

\[
f''(x) \leq \frac{24x^2 - 8}{(1 + 0^2)^3} = 24x^2 - 8
\]

which is maximized when \( x = 1 \), so \( f''(x) \leq 16 \).

Therefore, since \( f''(c) \leq 16 \) on \([0, 1]\), \( E_M \leq \frac{16(1 - 0)^3}{24n^2} = \frac{2}{3n^2} \).
5. (8.8 #39) Show that \( \frac{1}{2}(T_n + M_n) = T_{2n} \)

Let \( P \) be a partition \( \{x_0, x_1, x_2, \ldots, x_{2n}\} \). To work on the left-hand side with only \( n \) subintervals, we only use the even-indexed numbers \( x_0, x_2, x_4, \) etc.

Then \( T_n = \frac{\Delta x}{2} (f(x_0) + 2f(x_2) + 2f(x_4) + \cdots + 2f(x_{2n-2}) + f(x_{2n})) \)

Since the midpoint of \([x_0, x_2]\) is \( x_1 \) and so on, we have
\[
M_n = \Delta x (f(x_1) + f(x_3) + f(x_5) + \cdots + f(x_{2n-1}))
\]

To combine \( T_n \) and \( M_n \), note that
\[
M_n = \Delta x \left( 2f(x_1) + 2f(x_3) + 2f(x_5) + \cdots + 2f(x_{2n-1}) \right),
\]

so
\[
T_n + M_n = \Delta x (f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{2n-2}) + 2f(x_{2n-1}) + f(x_{2n})).
\]

With \( n \) subintervals, \( \Delta x = \frac{b - a}{n} \), so
\[
T_n + M_n = \frac{(b - a)}{2n} (f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{2n-2}) + 2f(x_{2n-1}) + f(x_{2n}))
\]

However, with \( 2n \) subintervals on the right hand side, \( \Delta x = \frac{b - a}{2n} \). Multiplying by \( \frac{1}{2} \) gives us \( b - a = \frac{\Delta x}{2} \) for the right. Therefore,
\[
\frac{1}{2}(T_n + M_n) = \frac{(b - a)}{4n} (f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{2n-2}) + 2f(x_{2n-1}) + f(x_{2n}))
\]
which is \( T_{2n} \).