Math 142 Lecture Notes
Section 3.3 – The Derivative

Rate of Change:
Examples:

1) Traveling 65 mph, means a change in distance (65 miles) with respect to a given time period (1 hour).

2) A cold front is expected tonight and is measured in rate of change of degrees over a given time period. (-5° per hour).

3) The slope of a line is the rate of change of the y-values, related to the changes in the x-values. \[ \frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x} \]

4) The profit (in dollars) from the sale of t-shirts at Paula’s Palace is given by \[ P(x) = 50x - 0.05x^2. \]
   a. What is the profit when 100 t-shirts are made and sold?
   b. What is the profit when 500 t-shirts are made and sold?
   c. What is the average change in profit if they increased production from 100 t-shirts to 500 t-shirts, made and sold?

Average Rate of Change:

For \( y = f(x) \), the average rate of change from \( x = a \) to \( x = a + h \) is

\[ \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h} \quad h \neq 0 \]

This is called the difference quotient.

Example 1: A water balloon dropped from the top of the apartment complex will fall a distance of \( y \) feet in \( x \) seconds, as given by the formula (from physics):

\[ y = f(x) = 16x^2 \]
• Find the average velocity from $x = 3$ seconds to $x = 4$ seconds.

• Find the average velocity from $x = 3$ seconds to $x = (3 + h)$ seconds.

• Find the limit of the expression as $h \to 0$.

```
Slope of a Graph: The slope of the graph of $y = f(x)$ at the point $(a, f(a))$ is given by: \[ \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \] provided that the limit exists.

Note: The slope of the graph is also the slope of the tangent line at the point.
```

**Example 2:** Given $f(x) = x^2 - 6x + 12$

1. Find the slope of the secant line for $a=3$, and $h=1$.

2. Find the slope of the secant line for $a=3$ and $h$ any nonzero number.

3. Find the limit of the expression in part 2, above.

4. Interpret your answer.
Interpretations of the Derivative.

1. The slope of the tangent line.
2. The instantaneous rate of change.
3. Velocity.

Examples:

A. Find \( f'(x) \), using the limit definition of the derivative for the function,
\[
f(x) = 3x - x^2
\]

B. Find the slope of the graph of \( f(x) \) at \( x = 0, \; x = 3, \; x = -2 \).

C. Find \( f'(x) \), using the limit definition of the derivative for the function,
\[
f(x) = \sqrt{x + 4}
\]

D. What is the domain of \( f(x) \) in part C above?

What is the domain of \( f'(x) \)?
The existence of a derivative at $x = a$ depends on the existence of a limit at $x = a$.

$$f'(x) = \lim_{{h \to 0}} \frac{f(a + h) - f(a)}{h}$$

If the limit does not exist at $x = a$, we say the function $f$ is **nondifferentiable at** $x = a$, or $f'(a)$ does not exist.

1. If the graph has a hole or a break at $x = a$, then $f'(a)$ does not exist.

2. If the graph of $f$ has a sharp corner at $x = a$, then $f'(a)$ does not exist.

3. If the graph of $f$ has a vertical tangent line at $x = a$, $f'(a)$ does not exist.