Math142 Lecture Notes
4.2 - Second Derivative and Graphs

<table>
<thead>
<tr>
<th>First Derivative</th>
<th>Second Derivative</th>
<th>$n^{th}$ derivative</th>
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</thead>
<tbody>
<tr>
<td>$f'(x)$</td>
<td>$f''(x)$</td>
<td>$f^{(n)}(x)$</td>
</tr>
<tr>
<td>$\frac{dy}{dx}$</td>
<td>$\frac{d^2y}{dx^2}$</td>
<td>$\frac{d^n y}{dx^n}$</td>
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Example 1: Find the first three derivatives for $f(x) = 2x^3 - 6x^2 - 48x + 20$.

Example 2: Discuss the difference in the shapes of the graphs of $f(x) = x^2$ and $g(x) = \sqrt{x}$.

<table>
<thead>
<tr>
<th>Clues to Graphing Functions</th>
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<tbody>
<tr>
<td>If $f(x) &gt; 0$, it’s graph lies above the x-axis.</td>
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<tr>
<td>If $f'(x) &gt; 0$, the graph of $f(x)$ is increasing.</td>
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<tr>
<td>If $f''(x) &gt; 0$, the graph of $f(x)$ is concave up.</td>
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Example 3: Given that $f(x) = x^3 - 6x^2 + 5$, determine the intervals where the graph:

a. is concave upward;

b. is concave downward.
Example 4: Given $f''(x) = 30x + 8$, find where $f''(x) = 0$ then find the intervals where $f'(x)$ is increasing and decreasing.

\[
\begin{align*}
\text{Concavity} \\
\text{On an open interval } (a, b) \text{ where } f \text{ is differentiable:} \\
1. \text{ If } f'(x) \text{ is increasing } (f''(x) > 0) \text{ then the graph of } f(x) \text{ is concave up.} \\
2. \text{ If } f'(x) \text{ is decreasing } (f''(x) < 0) \text{ then the graph of } f(x) \text{ is concave down.}
\end{align*}
\]

Example 5: Determine the intervals where the graph of $f(x) = x^3 + 3x^2 - 4$ is concave up and concave down.
Second Derivative Test (determines inflection values and concavity)

1. Find all inflection values. These are the values \( x = d \) where \( f''(d) = 0 \) or \( f''(d) \) is undefined.

2. Place these values on a number line and use the second derivative to generate a sign chart.

3. The point \((d, f(d))\) is an inflection point if \( f''(x) \) changes sign at \( x = d \) and if \( x = d \) is in the domain of \( f(x) \).

Example 6: Find the inflection points of \( f(x) \) if \( f''(x) = (x + 2)(x - 1)x^2 \).

Example 7: The Chug-a-Mug Company has determined that its cost, in hundreds of dollars, for producing \( x \) items of its best selling product is given by \( C(x) = x^3 - 6x^2 + 15x \).

(a) Determine \( C(5) \) and \( C'(5) \) and interpret each.

(b) Determine the intervals where \( C'(x) \) is increasing and where it is decreasing. Determine the relative minimum for the marginal cost function.

(c) Determine the inflection point for the graph of \( C \).