Domain decomposition preconditioners for multiscale flows in high contrast media

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Abstract

In this paper, we study domain decomposition preconditioners for multiscale flows in high-contrast media. We consider flow equations governed by elliptic equations in heterogeneous media with a large contrast in the coefficients. Our main goal is to develop domain decomposition preconditioners with the condition number that is independent of the contrast when there are variations within coarse regions. This is accomplished by designing coarse-scale spaces and interpolators that represent important features of the solution within each coarse region. The important features are characterized by the connectivities of high-conductivity regions. To detect these connectivities, we introduce an eigenvalue problem that automatically detects high-conductivity regions via a large gap in the spectrum. A main observation is that this eigenvalue problem has a few small, asymptotically vanishing, eigenvalues. The number of these small eigenvalues is the same as the number of connected high-conductivity regions. The coarse spaces are constructed such that they span eigenfunctions corresponding to these small eigenvalues. These spaces are used within two-level additive Schwarz preconditioners as well as overlapping methods for Schur complement to design preconditioners. We show that the condition number of the preconditioned systems is independent of the contrast. More detailed studies are performed for the case when the high conductivity region is connected within coarse block neighborhoods. Our numerical experiments confirm the theoretical results presented in this paper.

1 Introduction

Subsurface flows are often affected by heterogeneities in a wide range of length scales. Moreover, the media properties often vary significantly that introduces an additional level of complexity. A high contrast in the media properties brings

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an additional small scale into the problem expressed as the ratio between low and high conductivity values. For example, it is very common to have several orders of magnitude of variations in the permeability field in natural porous formations. In addition, low or high conductivity regions can have complex geometry that can introduce connected barriers at very small scales in the flow.

In this paper, we study efficient techniques for solving flows in highly heterogeneous formations. It is common to use upscaled or multiscale models to solve flow and transport processes on the coarse grid [1, 3, 4, 5, 9, 21]. These approaches approximate the effects of the fine-scale features and attempt to capture these effects on a coarse grid via localized basis functions. In many cases, multiscale approaches can not provide an accurate approximation of the solution and one needs to solve for the fine-scale solution. The solution of the fine-scale equation is typically prohibitively expensive because of the small scales and high contrast in the conductivity field. For this reason, some type of preconditioning is needed to reduce the number of iterations required for solving the fine-scale system of equations.

We investigate domain decomposition preconditioners for flows in heterogeneous porous media. Domain decomposition methods use the solutions of small local problems and a coarse problem in constructing preconditioners for the fine-scale system. The number of iterations required by domain decomposition methods is adversely affected by the contrast in the media properties. It is known that if high and low conductivity regions can be encompassed within coarse-grid blocks such that the variation of the conductivity within each coarse region is bounded, domain decomposition preconditioners result to a system with the condition number independent of the contrast (e.g., [18, 25]). Because of complex geometry of fine-scale features, it is often impossible to separate low and high conductivity regions into different coarse grid blocks.

The main component of our studies is to find coarse-scale spaces and interpolations that satisfy weighted $L^2$ approximation and stability estimates (see (11) and (12)). This guarantees that two-level additive Schwarz preconditioners converge independent of the contrast. First, we consider the case of locally connected high-conductivity regions. In this case, a coarse-scale interpolation can be defined using weighted averages. We show that the condition number in this case is independent of the contrast and one needs only one function to represent high-conductivity region within a coarse-grid block. In general cases, there may be many disconnected high-conductivity regions within each coarse-grid block and thus one needs larger coarse-scale spaces to represent them. To construct coarse spaces, we consider weighted eigenvalue problem with zero Neumann boundary conditions. This eigenvalue problem is motivated by weighted Poincaré estimates. It can be shown in the presence of high-contrast components within a coarse block, this local eigenvalue problem detects the disconnected high-conductivity regions through a large gap in the spectrum. In particular, there are a few eigenvalues that vanish as the number of contrast increases. The number of these eigenvalues is the same as the number of disconnected high-conductivity regions within the coarse-grid block. We prove that if the coarse space in domain decomposition methods includes the eigenfunctions cor-
responding to small, asymptotically, vanishing eigenvalues, then the condition number of the preconditioned matrix is bounded independent of the contrast. We use partition of unity functions to span the eigenfunctions. We note that the idea of trying to include spectral information in the coarse space is not new but up to our knowledge it has not been studied in the presence of contrast, e.g., [23, 24, 7, 8]. Our construction of the coarse space automatically selects appropriate basis elements independent of the geometry or scale separation in the problem.

In order to minimize the condition number of the preconditioned system, we also study overlapping domain decomposition preconditioners for the Schur complement problem. The Schur complement is obtained after eliminating the subdomain interior degrees of freedoms. We use an overlapping two-level additive method for the Schur complement problem ([2, 27]). Using this framework, we discuss how one can reduce the number of low conductivity inclusions using extension theorems [14, 10]. This reduction can be regarded as a partial homogenization because it homogenizes many inclusions such that the resulting media has fewer inclusions where our new Poincaré estimates can be applied.

Numerical results are presented to show that the condition number of the preconditioned system is independent of the contrast. In particular, we implement an approach where an arbitrary number of basis functions per node can be chosen. For the coarse solvers, we consider various choices for basis functions - piecewise linear basis, multiscale finite element basis functions, energy minimizing basis functions, and new coarse spaces obtained using the solutions of local weighted eigenvalue problem. We show that using appropriate coarse spaces one achieves less number of iterations and moreover, the condition number is bounded in contrast when other coarse spaces are used.

The paper is organized in the following way. In the next section, we present problem setting. Section 3 is devoted to the studies of two-level additive Schwarz preconditioners. In Section 4, we present numerical results. In Section 5, overlapping methods for Schur complement are presented. Finally, concluding remarks are presented in Section 6.

2 Problem Setting

Let \( D \subset \mathbb{R}^2 \) (or \( \mathbb{R}^3 \)) be a polygonal domain which is the union of a disjoint polygonal subregions \( \{D_i\}_{i=1}^N \). We consider the following problem. Find \( u^* \in H^1_0(D) \) such that

\[
a(u^*, v) = f(v) \quad \text{for all } v \in H^1_0(D). \tag{1}
\]

Here the bilinear form \( a \) and the linear functional \( f \) are defined by

\[
a(u, v) = \int_D \kappa(x) \nabla u(x) \nabla v(x) \quad \text{for all } u, v \in H^1_0(D) \tag{2}
\]

and \( f(v) = \int_D f(x) v(x) \quad \text{for all } v \in H^1_0(D). \)
We denote by $V^h(D)$ the usual finite element discretization of piecewise linear continuous functions with respect to the fine triangulation $T^h$. Denote also by $V^h_0(D)$ the subset of $V^h(D)$ with vanishing values on $\partial D$. Similar notations, $V^h(\Omega)$ and $V^h_0(\Omega)$, are used for subdomains $\Omega \subset D$.

The Galerkin formulation of (1) is to find $u^* \in V^h_0(D)$ such that

$$a(u^*,v) = f(v) \quad \text{for all } v \in V^h_0(D),$$

or in matrix form

$$Au^* = b,$$

where for all $u,v \in V^h(D)$ we have $u^T A v = \int_D \kappa \nabla u \nabla v$ and $v^T b = \int_D f v$.

Also, let $A^{(i)}$ be the local matrices corresponding to the bilinear form (2), i.e., for all $u,v \in V^h(D_i)$ we have

$$u^T A^{(i)} v = \int_{D_i} \kappa \nabla u \nabla v.$$ 

### 3 Overlapping methods

We denote by $\{D'_i\}_{i=1}^N$ the overlapping decomposition obtained from the original nonoverlapping decomposition $\{D_i\}_{i=1}^N$ by enlarging each subdomain $D_i$ to

$$D'_i = D_i \cup \{x \in D, \text{dist}(x,D_i) < \delta\}, \quad i = 1, \ldots, N,$$

where dist is some distance function and let $V^0_i(D'_i)$ be the set of finite element functions with support in $D'_i$. We also denote by $R^T_i : V^h_0(D'_i) \rightarrow V^h$ the extension by zero operator.

Now we introduce a coarse triangulation $T^H$ and coarse basis functions $\{\Phi_i\}_{i=1}^{N_c}$, where $N_c$ is the number of coarse basis functions. Note that the number of basis functions can be larger than the number of coarse-grid blocks. In the general setting of domain decomposition solvers, this coarse level triangulation may be independent of the subdomain partition of the original domain $D$. In order to simplify the analysis, we assume that the coarse triangulation coincides with the nonoverlapping decomposition. Define

$$V_0 = \text{span}\{\Phi_i\}_{i=1}^{N_c},$$

where $\Phi_i$, $i = 1, \ldots, N_c$, are coarse-scale finite element basis functions. We use a two level additive preconditioner of the form

$$B^{-1} = R^T_0 A_0^{-1} R_0 + \sum_{i=1}^N R^T_i A_i^{-1} R_i,$$

where the local matrices are defined by

$$v^T A_i w = a(R^T_i v, R^T_i w) \quad \text{for all } v,w \in V^h_0(D'_i), \quad i = 1, \ldots, N,$$
and the coarse matrix \( A_0 = R_0 A R_0^T \) with \( R_0^T : V^h \rightarrow R_0 \) being the projection onto the coarse space \( V_0 \), i.e., more precisely, the columns of \( R_0^T \) are the nodal vectors corresponding to the coarse basis functions. See [25, 18]. The operator \( R_i \) restricts a nodal vector to its subvector corresponding to nodes in the interior of \( D_i \).

We denote by \( \{y_i\}_{i=1}^{N_\varepsilon} \) the vertices of the coarse mesh \( T^H \) and define the neighborhood of the node \( y_i \) by

\[
\omega_i = \bigcup \{K \in T^H; \ y_i \in K\}
\]  

(9)

and the neighborhood of the coarse element \( K \) by

\[
\omega_K = \bigcup \{\omega_j; \ y_j \in K\}.
\]  

(10)

Throughout, \( a \preceq b \) means that \( a \leq Cb \) where the constant \( C \) is independent of the mesh size and contrast.

### 3.1 Condition number estimate. General concept.

The analysis of the preconditioner described in (7) lies within the family of additive Schwarz methods. Abstract domain decomposition theory has been developed for additive Schwarz methods, [25, 18]. According to this theory, in order to estimate the condition number of the preconditioner operator with the preconditioner (7), we need to verify three key assumptions on the subspaces and bilinear form involved: stable decomposition, strengthened Cauchy inequality and stable local solvers. In Lemma 1, we verify that the stable decomposition holds in general if we have a suitable coarse interpolation operator. The second and third key assumptions can be verified as in the bounded coefficient case; see Chapter 3 in [25].

In this section, we assume that there is a coarse interpolation \( I : V \rightarrow V_0 \) such that for all \( v \in V^h \) we have

\[
\int_K \kappa |v - I_0 v|^2 \leq CH^2 \int_{\omega_K} \kappa |\nabla v|^2
\]  

(11)

and

\[
\int_K \kappa |\nabla I_0 v|^2 \leq C \int_{\omega_K} \kappa |\nabla v|^2,
\]  

(12)

Next, we show that (cf. [18]) under these assumptions the condition number of the preconditioned matrix depends on the constants in (11) and (12). In the next sections, we will show that with appropriate coarse spaces, the constant \( C \) above is independent of the contrast.

**Lemma 1** Let \( \kappa \) be such that (11) and (12) hold true. For all \( v \in V^h \), there exists a decomposition \( v = \sum_{i=0}^{N} v_i \), with \( v_i \in V_0^i(\Omega_i^*) \), \( i = 0, 1, 2, \ldots \), such that

\[
a(v_0, v_0) + \sum_{i=1}^{N} a(v_i, v_i) \leq C_0^2 a(v, v),
\]
where $C_0^2 \leq C(1 + (\frac{H}{\delta})^2)$.

**Proof.** Define $v_0 := I_0v_h$, where $I_0$ is a coarse interpolation and

$$v_i = \xi_i(v - v_0),$$

where $\{\xi_i\}$ is a partition of unity subordinated to the overlapping partition $\{D'_i\}$ and $|\nabla \xi_i| \leq \frac{1}{\delta}$, $i = 1, \ldots, N$. First we bound the energy of $v_i$.

$$a(v_i, v_i) = \int_{D'_i} \kappa|\nabla (\xi_i(v - v_0))|^2 \leq \int_{D'_i} \kappa \xi_i^2|\nabla (v - v_0)|^2 + \int_{D'_i} \kappa |\nabla \xi_i|^2 |v - v_0|^2$$

$$\leq \int_{D'_i} \kappa |\nabla (v - v_0)|^2 + \frac{1}{\delta^2} \int_{D'_i \setminus D_i} \kappa |v - v_0|^2$$

$$\leq \int_{D'_i} \kappa |\nabla v|^2 + \int_{D'_i} \kappa |\nabla v_0|^2 + \frac{1}{\delta^2} \int_{D'_i \setminus D_i} \kappa |v - v_0|^2. \quad (13)$$

Now we bound the last two terms of (13).

The second term in (13) can be bounded using (12) as follows

$$\int_{D'_i} \kappa |\nabla v_0|^2 \leq \sum_{K \cap D'_i \neq \emptyset} \int_K \kappa |\nabla v_0|^2 \leq C \sum_{K \cap D'_i \neq \emptyset} \int_{\omega_K} \kappa |\nabla v|^2,$$

where $\omega_K = \cup\{K' : K \cap K' \neq \emptyset\}$. To bound the third term of (13), we use (11) to get

$$\frac{1}{\delta^2} \int_{D'_i \setminus D_i} \kappa |v - v_0|^2 \leq \frac{1}{\delta^2} \sum_{K \cap D_i \neq \emptyset} \int_K \kappa |v - v_0|^2 \quad (14)$$

$$\leq C \left(\frac{H}{\delta}\right)^2 \sum_{K \cap D_i \neq \emptyset} \int_{\omega_K} \kappa |\nabla v|^2.$$

The bound for the energy $a(v_0, v_0)$ follows from (12). \[\blacksquare\]

**Remark 2** In the proof of Lemma 1, we use the simple estimate (14). We avoid the use of bounds similar to the small overlap trick [25, Lemma 3.10] (valid for bounded coefficients) or [15, Lemma 3.4] (valid when a special partition robustness indicator is bounded). Such bounds would improve the final condition number of the method. Whether we can apply the small overlap trick within our analysis will be subject of future research.

We have the following bound for the condition number. See [25, 18].
Corollary 3 Under the assumptions of Lemma 1 the condition number of the preconditioned operator $B^{-1}A$ with $B^{-1}$ defined in (7) is of order

$$\text{cond}(B^{-1}A) \leq C \left( 1 + \left( \frac{H}{\delta} \right)^2 \right)$$

with $C$ independent of the contrast $\eta$.

With a generous overlap $\delta = cH$, we get $\text{cond}(B^{-1}A) \lesssim 1$.

In the next two sections, we will discuss coarse-scale interpolations for which inequalities (11) and (12) hold true independent of the contrast of the media. Our first example will be a case with one locally connected high-conductivity region within coarse-grid blocks, i.e., the background high-conductivity region is connected when we restrict to any coarse-grid neighborhood. This will be extended to the case with multiple high-conductivity regions within coarse-grid block neighborhoods. For the latter, we will show that one needs coarse-scale spaces with larger dimensions.

### 3.2 Locally connected high-conductivity case

We first verify a weighted Poincaré inequality for the coefficients with singly connected high-conductivity case, where any two points in the background high-conductivity region $\Omega_0$ can be connected while remaining in $\Omega_0$. Within $\Omega_0$, we will assume multiple low-conductivity components.

**Lemma 4** Let $\kappa$ be defined by

$$\kappa(x) = \begin{cases} \eta_i, & x \in \Omega_i, \\ \eta_0, & x \in \Omega_0 = \Omega \setminus \bigcup_{i=1}^{M} \Omega_i, \end{cases}$$

where $\Omega_i$, $i = 0, \ldots, M$ are Lipschitz domains. Assume also that

$$|\eta_i|_{\Omega_i} \lesssim |\eta_0|_{\Omega_0}, \quad i = 1, \ldots, M.$$  

Then, for every $v \in H^1(\Omega)$ such that $\int_\Omega \kappa v = 0$ we have

$$\int_\Omega \kappa v^2 \leq MH^2 \int_\Omega |\nabla v|^2.$$  

The proof of this lemma is presented in Appendix A.

**Remark 5** We note that Lemma 4 holds for any inclusions $\Omega_i$, $i = 1, \ldots, M$ that has Poincaré property (such that (57) and (56) hold).

**Remark 6** In Lemma 4, $\kappa$ is assumed to be piecewise constants. These results can be extended to positive piecewise smooth coefficients such that the functions in each region $\Omega_i$ are of the form $\eta_i g_i(x)$, where $|g_i| \leq 1$. 
Next, we construct a coarse-scale interpolation for the locally connected case. Here select one basis function per coarse node ($N_c = N_v$) and we assume that $V_0$ is defined as in (6) with $\|\Phi_i\|_\infty \leq 1$, $i = 1, \ldots, N_c$, and that $\sum_{i=1}^{N_c} \Phi_i = 1$. Note that here we need only one basis function per coarse node. Define $I_0 : V^h \to V^H$ by

$$I_0 v = \sum_{i=1}^{N_c} \tau_i \Phi_i$$

where $\tau_i = \frac{1}{\int_{\omega_i} \kappa \int_{\omega_i} \kappa v}$ if $y_i \in D$, \hspace{1cm} (15)

and $\tau_i = 0$ if $y_i \in \partial D$. Here $\omega_i$ is the support of $\Phi_i$ (the union of coarse blocks with the node $y_i$). Recall the definition of $\omega_K$ in (10). We further classify the coarse blocks in interior ($\overline{K} \subset \Omega$) and boundary ($\partial K \cap \partial \Omega \neq \emptyset$). In order to simplify the presentation we assume that the coefficient $\kappa$ has no variation within the boundaries of coarse blocks. Later we will drop this assumption.

**Lemma 7** Assume $N_c = N_v$ and that $\{\Phi_i\}$ is a partition of unity subordinated to the covering $\{\omega_i\}$ and $\|\Phi_i\|_\infty \leq 1$, $i = 1, \ldots, N_c$. Let the coefficient $\kappa$ be such that its restriction to $\omega_K$ satisfy the assumption of Lemma 4 for all interior coarse block $K$. Then for all $v \in V^h$ we have (11) and (12) with $C = \max_K M_K$, where $M_K$ is the number of low-conductivity inclusions in $\omega_K$.

**Proof.** Recall that for each coarse basis functions we have $\|\Phi_i\|_\infty \leq 1$. We only need to consider interior coarse blocks. Note that for any interior coarse block $K$ we have for $\tau_i$ defined in (15), $|\tau_i| \leq \frac{1}{(\int_{\omega_i} \kappa \int_{\omega_i} \kappa v)^{\frac{1}{2}}}$ and

$$\int_K \kappa |I_0 v|^2 \leq \sum_{y_i \in K} \left( \frac{1}{\int_{\omega_i} \kappa \int_{\omega_i} \kappa v} \right) \int_K \kappa |\Phi_i|^2 \leq \sum_{y_i \in K} \left( \frac{1}{\int_{\omega_i} \kappa \int_{\omega_i} \kappa v} \right) \int_K \kappa \leq C \int_{\omega_K} \kappa v^2.$$ Using this we obtain

$$\int_K \kappa |v - I_0 v|^2 \leq 2 \int_K \kappa |v|^2 + 2 \int_K \kappa |I_0 v|^2 \leq C \int_{\omega_K} \kappa v^2.$$ Then, if we put $\hat{v} = v - \frac{1}{\int_{\omega_K} \kappa} \int_{\omega_K} v$ we obtain

$$\int_K \kappa |\hat{v} - I_0 \hat{v}|^2 \leq C \int_{\omega_K} \kappa \hat{v}^2 \leq M_K H^2 \int_{\omega_K} \kappa |\nabla v|^2,$$
where we have used Lemma 4 with $\Omega = w_K$. Now we prove (12). Using again Lemma 4, we obtain

$$\int_K \kappa |\nabla I_0 v|^2 = \int_K \kappa |\nabla I_0 \hat{v}|^2 \leq \sum_{y_i \in K} \left( \frac{1}{\omega_i} \kappa \int_{\omega_i} \kappa \hat{v}^2 \right) \int_K \kappa |\nabla \Phi_i|^2 \leq \sum_{y_i \in K} \left( \frac{1}{\omega_i} \kappa \int_{\omega_i} \kappa \hat{v}^2 \right) H_K^{-2} \int_K \kappa \leq M_K H_K^{-2} \int_{\omega_K} k \hat{v}^2 \leq M_K \int_{\omega_K} k |\nabla v|^2.$$

(16)

This finishes the proof.

Now, we highlight the differences between our analysis and the analysis presented in [15, 16]; see also [25, 18]. In the analysis presented in [15], the authors use standard volume averaging for $v_i$. Thus, in the analysis, only standard Poincaré estimates are needed. In our analysis because of the weighted average definition of $v_i$ in (15), we need to employ special Poincaré estimates. As our later discussions show that one can handle general heterogeneities using coarse-scale spaces motivated by weighted Poincaré estimates. In particular, we need to represent each connected high-conductivity region on the coarse grid that will lead to enriched spaces. In [15], only one basis function per coarse node is employed and with this approach one can handle general cases.

Remark 8 One can choose the basis $\Phi_i$ such that $\int_K \kappa |\nabla \Phi_i|^2$ is small provided $\sum_i \Phi_i = 1$ (as suggested in [15, 16]), e.g., multiscale finite element basis functions or energy minimizing basis functions. It is evident from (16) that this will reduce the constant in (16) that directly affects the condition number; see Lemma 1 and Corollary 3 below.

Remark 9 From Lemma 4, we see that the constant of Corollary 3 depends on the number of inclusions. In a subsequent section, we discuss how one can reduce the number of inclusions based on extension theorems.

The analysis presented in this section is valid for coefficients that in each $\omega_K$ are as in Lemma 4 (with $\Omega = \omega_K$). The analysis is also valid for more general coefficients; see Remark 6. For numerical examples, see Section 4.

3.3 General coefficients

3.3.1 High-contrast eigenvalue problem and weighted Poincare inequality

In this section, we consider a general case where there are multiple high-conductivity regions that are disconnected from (or do not communicate with) each other. In previous section, we showed that for the case of singly connected high-conductivity region, only one basis function per node is needed. When there are multiple high-conductivity regions, one needs to enrich the coarse spaces to
Figure 1: Eigenfunctions for Neumann problem. Permeability is $10^7$ in the inclusions and channels and 1 outside. From left to right: permeability, second, third, and fourth eigenfunctions. The four small eigenvalues are 0, $1.1e-5$, $2.3e-5$, $3.2e-5$.

get robust preconditioners as will be discussed next. Our goal is to find a coarse space and interpolator such that that (11) and (12) hold. To construct such spaces we explore spectral “gap” in the following eigenvalue problem

$$\text{div}(\kappa(x)\nabla \phi_i) = \lambda_i \kappa(x)\phi_i. \tag{17}$$

This eigenvalue problem is solved in a union of coarse-grid blocks with a common vertex. It turns out that the problem (17) has eigenvalues that scale as the inverse of the high contrast. In particular, the number of eigenvalues that scale as the inverse of high conductivity is the same as the number of connected high-conducting regions. E.g., in Figure 1, we plot eigenvectors for a case with four inclusions two channels and two inclusions that have high conductivity. The background conductivity is one. As a result, there are four eigenvalues that are small and asymptotically vanish as the high conductivity increases. These eigenvalues are scaled as the inverse of the high contrast and the rest of eigenvalues are large and remain bounded below as the contrast increases. Assume that the eigenvalues are ordered as $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_L < \lambda_{L+1} \leq \ldots \leq \lambda_N$, where $\lambda_1, \ldots, \lambda_L$ are small, asymptotically vanishing eigenvalues. We show that if the coarse space includes the eigenfunctions corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_L$, then the condition number of the preconditioned matrix scales as $\max(1/\lambda_{L+1})$, where maximum is taken over all coarse grid blocks. This result holds in general and depends on which eigenvectors are included in the coarse space. Once the eigenfunctions corresponding to small, asymptotically vanishing, eigenvalues are selected in the coarse space, one can show that the condition number of the preconditioned system is bounded independent of the contrast.

Next, we introduce some notations for local eigenvalue problem to define coarse space. For any $\Omega \subset D$, we define the Neumann matrix $A^\Omega$ by

$$v^T A^\Omega w = \int_{\Omega} \kappa \nabla v \cdot \nabla w \quad \text{for all} \ v, w \in \tilde{V}^h(\Omega), \tag{18}$$

and the mass matrix of same dimension $M^\Omega$ by

$$v^T M^\Omega w = \int_{\Omega} \kappa vw \quad \text{for all} \ v, w \in \tilde{V}^h(\Omega), \tag{19}$$

where

$$\tilde{V}^h(\Omega) = \begin{cases} V^h(\Omega), & \text{if } \overline{\Omega} \cap \partial D = \emptyset \\ V^h(\Omega) : v = 0 \text{ on } \partial \overline{\Omega} \cap \partial D, & \text{otherwise}. \end{cases} \tag{20}$$
We consider the finite dimensional symmetric eigenvalue problem
\[ A^\Omega \phi = \lambda M^\Omega \phi, \quad \phi \in \tilde{V}_h(\Omega) \] (21)
and denote its eigenvalues and eigenvectors by \{\lambda^\Omega_\ell\} and \{\psi^\Omega_\ell\}, respectively. Note that the eigenvectors \{\psi^\Omega_\ell\} form an orthonormal basis of \(V^h(\Omega)\) with respect to the \(M^\Omega\) inner product. Assume that
\[ \lambda^\Omega_1 \leq \lambda^\Omega_2 \leq \cdots \leq \lambda^\Omega_i \leq \cdots, \] (22)
and note that \(\lambda^\Omega_1 = 0\). The eigenvalue problem above corresponds to the approximation of the eigenvalue problem \(\text{div}(\kappa \nabla u) = \lambda \kappa u\) in \(\Omega\) with Neumann boundary condition. In particular, \(\psi^\Omega_i\) denotes the \(i\)-th eigenvector of the Neumann matrix associated to the neighborhood of \(y_i\). If there are \(n\) inclusions and channels, then one can observe \(n\) small, asymptotically vanishing, eigenvalues. The eigenvectors corresponding to these eigenvalues will be used to construct the coarse space \(V_0\). We note that for the proposed methods, we only need to specify the eigenvectors based on the quantities \(\{1/\lambda^\omega_i\}\) in each \(\omega_i\). These eigenvectors are used to construct the coarse space.

We assume that the elements of \(T^h\) contained in \(\Omega\) form a triangulation of \(\Omega\). Let \(n_h(\Omega)\) denote the number of degrees of freedom in \(\Omega\). Given any \(v \in V^h(\Omega)\) we can write
\[ v = \sum_{\ell=1}^{n_h(\Omega)} (v^T M^\Omega \psi^\Omega_\ell) \psi^\Omega_\ell = \sum_{\ell=1}^{n_h(\Omega)} \left( \int_\Omega \kappa v \psi^\Omega_\ell \right) \psi^\Omega_\ell \]
and compute
\[ \int_\Omega \kappa |\nabla v|^2 = v^T A^\Omega v = \sum_{\ell=1}^{n_h(\Omega)} \left( \int_\Omega \kappa v \psi^\Omega_\ell \right)^2 \lambda^\Omega_\ell \] (23)
and
\[ \int_\Omega \kappa v^2 = v^T M^\Omega v = \sum_{\ell=1}^{n_h(\Omega)} \left( \int_\Omega \kappa v \psi^\Omega_\ell \right)^2. \] (24)

Given an integer \(L\) and \(v \in V^h(\Omega)\), we define
\[ I^\Omega_L v = \sum_{\ell=1}^L \left( \int_\Omega \kappa v \psi^\Omega_\ell \right) \psi^\Omega_\ell. \] (25)

From (22), (23), and (24) it is easy to prove the following inequality
\[ \int_\Omega \kappa(v - I^\Omega_L v)^2 \leq \frac{1}{\lambda^\Omega_{L+1}} a(v - I^\Omega_L v, v - I^\Omega_L v) \leq \frac{1}{\lambda^\Omega_{L+1}} a(v, v). \] (26)
When \(L = 1\) and \(\kappa = 1\) we obtain the usual Poincaré inequality since it can be verified that \(\lambda^\Omega_2 = O(\text{diam}(\Omega)^{-2})\), where \(\text{diam}(\Omega)\) is the diameter of \(\Omega\). If \(L\) is large enough then \(\lambda^\Omega_{L+1}\) is contrast independent and in this case we refer to (26) as a contrast independent weighted Poincaré inequality.
3.3.2 Coarse-scale interpolation operator and condition number estimate

We note that \( \{ \omega_i \}_{y_j \in \mathcal{T}^h} \) is a covering of \( \Omega \). Let \( \{ \chi_i \}_{i=1}^{N_c} \) be a partition of unity subordinated to the covering \( \{ \omega_i \} \) such that \( \chi_i \in V^h(D) \) and \( |\nabla \chi_i| \leq \frac{1}{H} \), \( i = 1, \ldots, N_c \). Define the set of coarse basis functions

\[
\Phi_{i,\ell} = I^h(\chi_i \psi_{\omega_i}^{\ell}) \quad \text{for } 1 \leq i \leq N_c \text{ and } 1 \leq \ell \leq L_i, \tag{27}
\]

where \( I^h \) is the fine-scale nodal value interpolation and \( L_i \) is an integer number for each \( i = 1, \ldots, N_c \). Note that in this case, there are several basis functions per coarse node. The number of basis functions per node is defined via an eigenvalue problem. Denote by \( V_0 \) the local spectral multiscale space

\[
V_0 = \text{span}\{ \Phi_{i,\ell} : 1 \leq i \leq N_c \text{ and } 1 \leq \ell \leq L_i \}. \tag{28}
\]

Define also the coarse interpolation \( I_0 : V^h(D) \to V_0 \) by

\[
I_0 v = \sum_{i=1}^{N_c} \sum_{\ell=1}^{L_i} \left( \int_{\omega_i} \kappa v \psi_{\omega_i}^{\ell} \right) I^h(\chi_i \psi_{\omega_i}^{\ell}) = \sum_{i=1}^{N_c} I^h \left( (I^h_{\omega_i} v) \chi_i \right), \tag{29}
\]

where \( I^h \) is the fine-scale nodal value interpolation and \( I^h_{\omega_i} \) is defined in (25).

Since \( \psi_{\omega_i}^{\ell} \in \tilde{V}_h(\omega_i) \) we see that \( I_0 v \) satisfies the zero Dirichlet boundary condition on \( \partial D \). Note that we have

\[
v - I_0 v = \sum_{i=1}^{N_c} I^h \left( \chi_i (v - I^h_{\omega_i} v) \right) .
\]

We have the weighted \( L^2 \) approximation and weighted \( H^1 \) stability properties similar to those in (11) and (12).

**Lemma 10** For all coarse element \( K \) we have

\[
\int_K \kappa (v - I_0 v)^2 \leq \frac{1}{\lambda_{K,L+1}} \int_{\omega_K} \kappa |\nabla v|^2, \tag{30}
\]

\[
\int_K \kappa |\nabla I_0 v|^2 \leq \max\{1, \frac{1}{H^2 \lambda_{K,L+1}} \} \int_{\omega_K} \kappa |\nabla v|^2, \tag{31}
\]

where \( \lambda_{K,L+1} = \min_{y_j \in K} \lambda^2_{K,L+1} \) and \( \omega_K \) is the union of the elements that share common edge with \( K \) defined in (10).

**Proof.** First we prove (30). Using (68) (see Appendix B) and the fact that
\[ \chi_i \leq 1 \] we have
\[
\int_K \kappa(v - I_0 v)^2 \leq \sum_{y_i \in K} \int_K \kappa I^h_i (\chi_i (v - I^\omega_{L_i} v)^2 \quad (32)
\]
\[
\leq \sum_{y_i \in K} \int_K \kappa(\chi_i (v - I^\omega_{L_i} v))^2 \quad (33)
\]
\[
\leq \sum_{y_i \in K} \int_{\omega_i} \kappa(v - I^\omega_{L_i} v)^2 \quad (34)
\]
and using (26) with \( \Omega = \omega_i \) to estimate the last term above, we obtain
\[
\int_K \kappa(v - I_0 v)^2 \leq \sum_{y_i \in K} \frac{1}{\lambda^2_{L+1}} \int_{\omega_i} \kappa |\nabla v|^2 \quad (35)
\]
\[
\leq \max_{y_i \in K} \frac{1}{\lambda^2_{L+1}} \int_{\omega_K} \kappa |\nabla v|^2. \quad (36)
\]
To prove the stability (31) we note that in \( K \sum_{y_i \in K} \nabla \chi_i = 0 \), and then we can fix \( y_j \in K \) and write \( \nabla \chi_j = -\sum_{y_i \in K \setminus \{y_j\}} \nabla \chi_i \). We obtain,
\[
\nabla \sum_{y_i \in K} (I^\omega_{L_i} v) \chi_i = \sum_{y_i \in K} \nabla \chi_i (I^\omega_{L_i} v) + \sum_{y_i \in K} \chi_i \nabla (I^\omega_{L_i} v) \quad (37)
\]
\[
= \sum_{y_i \in K \setminus \{y_j\}} (I^\omega_{L_i} v - I^\omega_{L_j} v) \nabla \chi_i + \sum_{y_i \in K} \chi_i \nabla (I^\omega_{L_i} v) \quad (38)
\]
which gives the following bound valid on \( K \),
\[
|\nabla \sum_{y_i \in K} (I^\omega_{L_i} v) \chi_i|^2 \leq \frac{1}{H^2} \sum_{y_i \in K \setminus \{y_j\}} (I^\omega_{L_i} v - I^\omega_{L_j} v)^2 + \sum_{y_i \in K} |\nabla (I^\omega_{L_i} v)|^2. \quad (39)
\]
Since \( \sum_{y_i \in K} (I^\omega_{L_i} v) \chi_i \in \mathbb{P}^3(K) \) we can use (68) and (39) to get
\[
\int_K \kappa |\nabla I_0 v|^2 = \int_K \kappa |\nabla I^h_i (\sum_{y_i \in K} (I^\omega_{L_i} v) \chi_i)|^2 \quad (40)
\]
\[
\leq \int_K \kappa |\nabla \sum_{y_i \in K} (I^\omega_{L_i} v) \chi_i|^2 \quad (41)
\]
\[
\leq \sum_{y_i \in K} \frac{1}{H^2} \int_K \kappa (I^\omega_{L_i} v - I^\omega_{L_j} v)^2 + \sum_{y_i \in K} \int_K \kappa |\nabla (I^\omega_{L_i} v)|^2. \quad (42)
\]
To bound the first term above we use (26) with \( \Omega = \omega_i \) as follows,
\[
\int_K \kappa (I^\omega_{L_i} v - I^\omega_{L_j} v)^2 \leq \int_{\omega_i} \kappa (v - I^\omega_{L_i} v)^2 + \int_{\omega_j} \kappa (v - I^\omega_{L_j} v)^2 \quad (43)
\]
\[
\leq \frac{1}{\lambda^2_{L+1}} \int_{\omega_i} \kappa |\nabla v|^2 + \frac{1}{\lambda^2_{L+1}} \int_{\omega_j} \kappa |\nabla v|^2 \quad (44)
\]
\[
\leq \frac{1}{\lambda L+1} \int_{\omega_K} \kappa |\nabla v|^2. \quad (45)
\]
The second term in (42) is estimated using (23) and the orthogonality of the eigenvectors in the $A^{-1}$ inner product

$$
\int_K \kappa|\nabla (I_{L_i}^\omega v)|^2 \leq \int_{\omega_i} \kappa|\nabla (I_{L_i}^\omega v)|^2 \leq \int_{\omega_i} \kappa|\nabla v|^2 \leq \int_{\omega_K} \kappa|\nabla v|^2.
$$

(46)

By combining (45), (46) and (42) we obtain (31).

**Corollary 11** Under the assumptions of Lemma 10, the condition number of the preconditioned operator $B^{-1}A$ with $B^{-1}$ defined in (7) is of order

$$
\text{cond}(B^{-1}A) \leq C_0^2 \leq \max\{1 + \frac{1}{\delta^2 \lambda_{L+1}}, 1 + \frac{1}{H^2 \lambda_{L+1}}\},
$$

where $\lambda_{L+1} = \min_{1 \leq i \leq N_c} \lambda_{L+1}^{\omega_i}$. 

**Remark 12** Assume that $\kappa(x) = 1$ for all $x \in D$ and $L_i = 1$ for all $i = 1, \ldots, N_c$. Then (30) and (31) become

$$
\int_K \kappa(v - I_0 v)^2 \leq H^2 \int_{\omega_K} \kappa|\nabla v|^2 \leq \int_{\omega_K} \kappa|\nabla I_0 v|^2 \leq \int_{\omega_K} \kappa|\nabla v|^2,
$$

respectively, because $\lambda_{K,L+1} = \max_{y_i \in K} \lambda_{L+1}^{\omega_i} \approx H^{-2}$.

**Corollary 13** Under the assumptions of Lemma 1 the condition number of the preconditioned operator $B^{-1}A$ with $B^{-1}$ defined in (7) is of order

$$
\text{cond}(B^{-1}A) \leq C(1 + \frac{H^2}{\delta^2}),
$$

where $C$ is independent of the contrast and the mesh size.

### 3.3.3 Energy minimizing partition of unity

Coarse-scale spaces with small energy often reduce the condition number. One can choose partition of unity functions $\{\chi_i\}_{i=1}^{N_c}$ such that they provide the least energy. This can be accomplished by solving

$$
\min \sum_{i=1}^{N_c} \sum_{\ell=1}^{L_i} \int_{\omega_i} \kappa|\nabla \Phi_{i,\ell}|^2 = \min \sum_{i=1}^{N_c} \sum_{\ell=1}^{L_i} \int_{\omega_i} \kappa|\nabla (\chi_i \psi_{i,\ell}^{(\omega_i)})|^2,
$$

(47)

where the local spectral multiscale basis functions $\Phi_{i,\ell}$ are defined in (27) and the minimum is taken over all partition of unity functions $\{\chi_i\}_{i=1}^{N_c}$ subordinated to the covering $\{\omega_i\}_{i=1}^{N_c}$ of $D$. In [26], the authors considered energy minimizing partition of unity basis for $\kappa(x)$. This is the same if we take one constant unity eigenfunction in our formulation. One can solve (47) following the similar procedure as discussed in [26].

The numerical solution of the energy minimizing problem (47) is computationally intensive and requires computing a global problem. One can use instead multiscale basis functions that share similarities with energy minimizing basis functions. In particular, they reduce the energies of the basis functions.
In this section, we consider the multiscale framework that can compute basis functions with less energy by solving one local problem per coarse block.

We define the \textit{local spectral multiscale basis functions with reduced energy} $\tilde{\Phi}_{i,\ell}$ as the $\kappa$--harmonic extension of $\Phi_{i,\ell}$ in each coarse block, that is, for each coarse element $K$ and $1 \leq \ell \leq L_i$ with $1 \leq i \leq N_c$ we have

$$
\int_K \kappa \nabla \tilde{\Phi}_{i,\ell} \nabla z = 0 \quad \text{for all} \quad z \in V_0^h(K), \quad (48)
$$

$$
\tilde{\Phi}_{i,\ell} = \Phi_{i,\ell} \quad \text{on} \quad \partial K.
$$

We define \textit{local spectral multiscale coarse space with reduced energy} by

$$
V_0 = \text{span}\{\tilde{\Phi}_{i,\ell}\}_{i=1}^{N_c}.
$$

These basis functions (47) or (48) can be used to reduce the dimension of the coarse space. This will be discussed elsewhere.

We would like to note that the choice of the coarse grid is important in designing coarse spaces. In some situations, with an appropriate choice of the coarse grid, one can ensure that each coarse region contains singly connected high-conductivity region, thus, there is only one coarse basis function per node is needed. However, this may not always be possible or it can impose strong restrictions on the coarse grid, e.g., can result to very large-size coarse regions. In this paper, we do not discuss how to choose the coarse grid, instead we focus on designing coarse spaces for a given coarse-grid geometry. As we show that one needs to represent each disconnected high-conductivity region within a coarse block. In Section 5.2, we discuss how to remove the isolated inclusions and reduce the dimension of the coarse space.

4 Numerical experiments

We solve the equation $-\text{div}(\kappa \nabla u) = 0$ with $u(x_1, x_2) = x_1$ on $\partial D$. We run the Preconditioned Conjugate Gradient (PCG) until the $\ell_2$ norm of the residual is reduced by a factor of $10^6$. We will present the number of iterations until the convergence of the PCG because this quantity is a main factor in determining the efficiency of the preconditioner. We have also tested our methods for $-\text{div}(\kappa \nabla u) = f$ in $D$ and $u = 0$ on $\partial D$ and observed similar results, i.e., preconditioned system is independent of the contrast. For our first numerical example, we consider the coefficients where there is only one connected high-conductivity region, see Figure 4 (left). In this case, we expect one basis function per coarse node will be sufficient for constructing contrast-independent preconditioners. The domain $D = [0, 1] \times [0, 1]$ that is divided into $10 \times 10$ equal square subdomains. Inside each subdomain we use a fine-scale triangulation where triangular elements constructed from $10 \times 10$ squares are used. We test different values for the contrast in the numerical examples. We implement two level additive preconditioner with three of the classical coarse spaces: $P^1$--linear functions (Linear), multiscale functions with linear boundary condition (MS) and energy
minimizing functions (EMF). In Table 1 we show the iteration count for the two level additive methods for (4) and (51). We tested different coarse basis: piecewise linear, multiscale basis and energy minimizing functions as defined in [26]. Note that the coarse triangulation is not aligned with the inclusions in the numerical test. We observe from this table that the condition number of the resulting preconditioned system is independent of the contrast as the theory indicates for singly connected high-conductivity case. We observe slight improvement when multiscale and energy minimizing basis functions are used for the coarse-scale approximation.

Figure 2: $\kappa(x)$ illustration. Red designates the regions where the coefficient is $\eta$ and blue designates the regions where the coefficient is 1.

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>Linear</th>
<th>MS</th>
<th>EMF</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>13</td>
<td>13</td>
<td>18</td>
</tr>
<tr>
<td>10</td>
<td>18</td>
<td>15</td>
<td>21</td>
</tr>
<tr>
<td>100</td>
<td>21</td>
<td>17</td>
<td>22</td>
</tr>
<tr>
<td>1000</td>
<td>21</td>
<td>18</td>
<td>22</td>
</tr>
<tr>
<td>10000</td>
<td>21</td>
<td>18</td>
<td>23</td>
</tr>
<tr>
<td>100000</td>
<td>21</td>
<td>18</td>
<td>23</td>
</tr>
<tr>
<td>1000000</td>
<td>22</td>
<td>18</td>
<td>23</td>
</tr>
</tbody>
</table>

Table 1: Number of iteration until convergence of the PCG for different values of the contrast $\eta$ see Figure 4 (left figure). We solve (4) with preconditioner $B^{-1}$ in (7). Different coarse problems: piecewise linear (Linear), multiscale (MS) and energy minimizing functions (EM). Here we have $H = 1/10$, $h = 1/100$ and $\delta = H$.

Next, we consider a case with multiple high-conductivity regions that are disconnected. In particular, we interchange high and low conductivities of the permeability considered in the previous example, Figure 4 (right plot) and consider the same fine- and coarse-grid settings. In this case, we need larger coarse spaces for constructing contrast-independent preconditioners. We use the additive preconditioner (7) with the local spectral multiscale coarse space defined in (28). The definition of the local spectral multiscale coarse space $V_0$ in (28) depends on the choice of a partition of unity. In the first numerical experiments
Table 2: Number of iterations until convergence of the PCG and condition number for different values of the contrast $\eta$ with the coefficient depicted in Figure 4 (right plot). We set the tolerance to $1e-10$. Here $H = 1/10$ with $h = 1/100$. The classical (one basis per node) coarse problems size is $81 \times 81$. The spectral coarse problem is of size $321 \times 321$. We will use two partition of unity functions: $P^1$ bilinear functions ($LSM_1$) and usual multiscale functions (see [19]) with linear boundary conditions ($LSM_2$). In Table 2 we set

$$L_i = L = \max_{\lambda \leq 2} \ell$$

(50) for all $i = 1, \ldots, N_c$. We note that $L_i \leq 4$ for all $i = 1, \ldots, N_c$ in this case. In Table 2 we present the number of iterations until convergence and in parenthesis the conjugate gradient estimate for the condition number of the preconditioned operator for the methods mentioned above. The results show an agreement with our theory. We observe that for the classical coarse spaces (with one basis per coarse node), the number of iterations and the condition number depend on the contrast ($\eta$) while for the spectral coarse spaces (last two columns) the number of iterations and the condition number remains bounded as the contrast increases. The dimension of the classical coarse matrix is $81 \times 81$ and the dimension of the spectral coarse problem is $321 \times 321$. This is even smaller than the dimension of one local problem $400 \times 400$.

Next, we consider coefficients that contain high-conductivity channels and inclusions as depicted in Figure 3. This corresponds to the case with the background conductivity one and broken channels with high conductivity $\eta$. We present numerical results using local spectral coarse space $V_0$ with reduced energy defined in (48) and (49). We consider $D = [0,1] \times [0,1]$ that is divided into $8 \times 8$ equal square subdomains with a fine scale triangular elements constructed from $10 \times 10$ squares. As before, we compare domain decomposition methods with spectral coarse spaces and domain decomposition methods with classical spaces: $P^1$–linear functions (Linear), multiscale functions with linear boundary condition (MS), energy minimizing functions (EMF). We also use two partition of unity functions: $P^1$ bilinear functions ($LSM_1$) and usual multiscale functions with linear boundary conditions ($LSM_2$). The local spectral multiscale coarse space with reduced energy (LSM-RE) uses the space $V_0$ defined in (49). The number of eigenvalues in each node is computed as before, see (50).

In Table 3 and 4 we present the number of iterations until convergence and in the parenthesis the conjugate gradient estimate for the condition number of the preconditioned operator. We observe that for the new coarse space (last three

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>Linear</th>
<th>MS</th>
<th>EMF</th>
<th>LSM$_1$</th>
<th>LSM$_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^3$</td>
<td>81(3.8e+3)</td>
<td>68(2.1e+3)</td>
<td>52(3.8e+2)</td>
<td>53(54.29)</td>
<td>46(32.04)</td>
</tr>
<tr>
<td>$10^5$</td>
<td>88(3.8e+4)</td>
<td>73(2.1e+4)</td>
<td>60(3.8e+3)</td>
<td>41(56.25)</td>
<td>40(32.73)</td>
</tr>
<tr>
<td>$10^6$</td>
<td>111(3.8e+5)</td>
<td>91(2.1e+5)</td>
<td>68(3.8e+4)</td>
<td>40(56.53)</td>
<td>37(33.40)</td>
</tr>
<tr>
<td>$10^7$</td>
<td>141(3.8e+6)</td>
<td>112(2.1e+6)</td>
<td>76(3.8e+5)</td>
<td>37(56.52)</td>
<td>36(41.35)</td>
</tr>
<tr>
<td>$10^8$</td>
<td>156(3.8e+7)</td>
<td>129(2.1e+7)</td>
<td>86(3.8e+6)</td>
<td>37(56.42)</td>
<td>33(42.47)</td>
</tr>
<tr>
<td>$10^9$</td>
<td>175(3.8e+8)</td>
<td>143(2.1e+8)</td>
<td>73(2.3e+7)</td>
<td>30(42.59)</td>
<td>30(42.59)</td>
</tr>
</tbody>
</table>
Table 3: Number of iterations until convergence of the PCG and condition number for different values of the contrast $\eta$ with the coefficient in Figure 3 (left plot). We set the tolerance to $1e^{-10}$. Here $H = 1/8$ with $h = 1/80$. The classical coarse space is of dimension $49 \times 49$ and the dimension of the new coarse space is $126 \times 126$.

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>Linear</th>
<th>MS</th>
<th>EMP</th>
<th>LSM_1</th>
<th>LSM_2</th>
<th>LSM-RE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^1$</td>
<td>92(3.5e+3)</td>
<td>77(2.3e+3)</td>
<td>57(364.12)</td>
<td>33(8.54)</td>
<td>32(8.40)</td>
<td>29(8.25)</td>
</tr>
<tr>
<td>$10^6$</td>
<td>109(3.4e+4)</td>
<td>93(2.2e+4)</td>
<td>58(404.85)</td>
<td>34(8.56)</td>
<td>34(8.42)</td>
<td>28(8.27)</td>
</tr>
<tr>
<td>$10^7$</td>
<td>124(3.4e+5)</td>
<td>107(2.3e+5)</td>
<td>67(409.46)</td>
<td>35(8.56)</td>
<td>34(8.42)</td>
<td>29(8.27)</td>
</tr>
<tr>
<td>$10^8$</td>
<td>144(3.4e+6)</td>
<td>137(2.3e+6)</td>
<td>77(409.93)</td>
<td>36(8.56)</td>
<td>35(8.42)</td>
<td>31(8.27)</td>
</tr>
<tr>
<td>$10^9$</td>
<td>163(3.4e+7)</td>
<td>159(2.3e+7)</td>
<td>82(409.97)</td>
<td>37(8.56)</td>
<td>36(8.41)</td>
<td>32(8.27)</td>
</tr>
</tbody>
</table>

columns) the number of iterations and the condition number remain bounded as the contrast increases. We see that the domain decomposition methods with local spectral coarse space with reduced energy (defined in (49)) performs slightly better than the local spectral coarse space (defined in (28)).

5 Overlapping methods for the Schur complement

We note that the number of low-conductivity inclusions adversely affect the condition number of the preconditioned operator. This is clear in the case of singly connected high-conductivity regions, see Corollary 3. One way to treat coefficient as in Lemma 4 with many inclusions with low conductivities (in each coarse neighborhood) is to use a Schur complement framework where we restrict to function that inside each coarse block neighborhood are harmonic extension of its coarse grid boundary values. To construct an overlapping method in the
Table 4: Number of iterations until convergence of the PCG and condition number for different values of the contrast $\eta$ with the coefficient in Figure 3 (right figure). We set the tolerance to $1e-10$. Here $H = 1/8$ with $h = 1/80$. The classical coarse space is of dimension $49 \times 49$ and the dimension of the new coarse space is $166 \times 166$.

Schur complement framework, we can then replace the original coefficient $\kappa$ by any coefficient $\kappa^*$ that gives equivalent energy in the subspace of functions that are harmonic extension of its coarse grid boundary values.

We present analysis of overlapping methods for Schur complement; see [2] and references therein. These methods employ harmonic extensions and allow us to use extension theorems (e.g., [14, 10]) to reduce the number of low conductivity inclusions. This reduction can be regarded as a partial homogenization.

### 5.1 Schur complement problem and corresponding preconditioners

We define the local interfaces $\Gamma_i = \partial D_i \cap D$, $i = 1, \ldots, N$ and the interface $\Gamma = \bigcup_{i=1}^{N} \Gamma_i$. Given $u \in V_h^0(D)$ we classify its degrees of freedom in: $u_{\Gamma}$, the interface degrees of freedom, and $u_I$, the interior degrees of freedom, those in the interior of any $D_i$ and outside $\Gamma$. The $2 \times 2$ block structure of the global matrix in (4) and the local matrices $A^{(i)}$, $i = 1, \ldots, N$ defined in (5) are

$$
\begin{bmatrix}
A_{II} & A_{I\Gamma} \\
A_{I\Gamma}^T & A_{\Gamma\Gamma}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\
A_{I\Gamma}^{(i)T} & A_{\Gamma\Gamma}^{(i)}
\end{bmatrix}.
$$

If we eliminate the interior variables in (4), we get linear system given by

$$
Su_{\Gamma} = \tilde{b},
$$

where $S = A_{\Gamma\Gamma} - A_{\Gamma I}^T A_{II}^{-1} A_{I\Gamma}$ and $\tilde{b} = b_{\Gamma} - A_{\Gamma I}^T A_{II}^{-1} b_I$. We note that applying $A_{II}^{-1}$ corresponds to the solution of $N$ uncoupled Dirichlet problems. This computation can be done in parallel. It can be seen that the above linear system corresponds to the problem: Find $u_\Gamma$ that satisfy:

$$
s(u_\Gamma, v_\Gamma) := a(\mathcal{H}_\kappa u_\Gamma, \mathcal{H}_\kappa v_\Gamma) = f(\mathcal{H}_\kappa v_\Gamma) \quad \text{for all } v \in V^\kappa(\Gamma),
$$

where $V^\kappa(\Gamma) = \{ v \in V_h^0(D) : v = \mathcal{H}_\kappa v_\Gamma \}$ and the $\kappa$–harmonic extension is defined by by

$$
a(\mathcal{H}_\kappa u, z) = 0 \quad \text{for all } z \in V_h^0(\Gamma), \mathcal{H}_\kappa u \big|_{\Gamma} = u \quad \text{on } \Gamma.
$$
Analogously, local Schur complement is given by $S^{(i)} = A_{IT}^{(i)} - A_{II}^{(i)} (A_{II}^{(i)})^{-1} A_{IT}^{(i)}$, $i = 1, \ldots, N$, and we can write

$$S = \sum_{i=1}^{N} R^{(i)T} S^{(i)} R^{(i)}, \quad (54)$$

where, for $i = 1, \ldots, N$, $R^{(i)T} : V^h(D_i) \cap V^0_0(D) \rightarrow V^0_0(D)$ is the extension by zero operator with a matrix which has only zeros and ones as entries. Note that here for simplicity we kept the notations for restrictions into the subdomains and the interfaces the same.

**Remark 14** We will solve the Schur complement system (52) or (51) using a preconditioned conjugate gradient method with two level Schwarz preconditioners. To apply $S$ we can use (54) which requires the solution of a local Dirichlet problem per subdomain.

Consider an overlapping decomposition $\{D'_i\}_{i=1}^{N'}$. We assume that this decomposition is aligned with the coarse triangulation, i.e., each $D'_i$ is the union of coarse blocks. Put $\Gamma'_i = \Gamma \cap D'_i$. Define the local spaces $V_i(\Gamma'_i)$, $i = 1, \ldots, N'$ as the function in $V^0_0(\Gamma)$ supported in $D'_i$. Let $S_i$ be the Schur complement of $A_i$ defined in (8) with respect to $\Gamma'_i$.

We use the following two level additive Schwarz preconditioners

$$M^{-1} = R^T S_0^{-1} R_0 + \sum_{i=1}^{N'} R_i^T S_i^{-1} R_i, \quad (55)$$

where $S_0 = (R_0 S R_0^T)^{-1}$ is the matrix form of $S$ on the coarse space $V_0$ define in (6) and $R^T_0$ is the projection on $V_0$. More precisely, the columns of $R^T_0$ are the nodal vectors corresponding to the coarse basis functions. Each application of $M^{-1}$ requires solving a coarse problem and a reduced local problem per subdomain.

The proof of the following Lemma 15 is similar to the proof of Lemma 1.

**Lemma 15** Let $\kappa$ be such that (30) and (31) in the space $V^0_0(\Gamma)$. For all $v \in V^0_0(\Gamma)$, there exists a decomposition $v = \sum_{i=0}^{N} v_i$, with $v_i \in V^i_0(\Gamma'_i)$, $i = 0, 1, 2, \ldots$, such that

$$s(v_0, v_0) + \sum_{i=1}^{N} s(v_i, v_i) \leq C_0^2 s(v, v),$$

where $C_0^2$ is independent of the contrast $\eta$.

Under the assumption of Lemma 15, the condition number of the preconditioned operator $M^{-1} S$ with $M^{-1}$ defined in (55) is bounded, i.e., $\text{cond}(M^{-1} S) \leq C$ with $C$ independent of the contrast $\eta$. 
5.2 Reducing the number of low-conductivity inclusions – partial energy homogenization

Previous analysis assumes that the number of low-conductivity inclusions is finite. The number of low-conductivity inclusions directly affects the number of iterations in domain decomposition methods as it is evident from Lemma 4. In many applications, the number of low-conductivity inclusions can be very large introducing a small scale into the problem. One can reduce the number of low-conductivity inclusions by using extension theorems (e.g., [14, 10]). This reduction can be regarded as a partial homogenization because it homogenizes many inclusions such that the resulting media have fewer inclusions and have an equivalent energy as the original media. This type of homogenization is possible within the Schur complement framework. We note that a difference from the homogenization is that the equivalent energy is sought for arbitrary boundary conditions here.

Next, we briefly describe it. Assume the domain Ω contains many inclusions with lower conductivities and denote these regions by Ω₁. As before Ω = Ω₁∪Ω₂, where Ω₁ denotes the union of domains that have conductivity 1. We call the media $E_\leq$ type if for any function $v \in H^1(\Omega_{\eta})$, there exists an extension $v_e$ such that $\int_\Omega |\nabla v_e|^2 \leq \int_{\Omega_{\eta}} |\nabla v|^2$. This holds for any inclusion $\Omega_1$ where $\Omega_{\eta}$ is a connected region [14]. It also holds for inclusions that are periodic or randomly distributed ([10]).

Next, we demonstrate how the extension theorem can be used to prove the energy equivalence. First, we show that if the domain is type $E_\leq$ then

$$\int_\Omega \kappa^*|\nabla H_{\kappa^*}\mu|^2 \leq \int_\Omega |\nabla H_{\kappa}\mu|^2 \leq \int_\Omega \kappa^*|\nabla H_{\kappa^*}\mu|^2$$

for all $\mu \in H^{1/2}(\partial\Omega)$, where $\kappa^* = \eta$ is the homogenized coefficient. Upper bound immediately follows because $\kappa \leq \kappa^*$. For the lower bound, we have

$$\int_\Omega \kappa^*|\nabla H_{\kappa^*}\mu|^2 \leq \int_{\Omega_{\eta}} \kappa^*|\nabla H_{\kappa^*}\mu|^2 \leq \int_\Omega \kappa^*|\nabla H_{\kappa^*}\mu|^2$$

where $v_e$ is the function that is equal to $v = H_{\kappa}\mu$ in $\Omega_{\eta}$ and its $E_\leq$ extension in $\Omega_1$.

The above estimate can be extended to more general case (see Figure 4), where $\Omega' \subset \Omega$ contains inclusions that can be homogenized using the extension theorem, while the rest of the domain $\Omega - \Omega'$ remains unchanged. By homogenizing a part of the region, we can lower the number of inclusions significantly, in general. The estimate for the upper bound follows in the same way as before. For the lower bound, we first use the variational principle and replace $H_{\kappa^*}\mu$ by $v_e$ and then divide the integral into two parts

$$\int_\Omega \kappa^*|\nabla H_{\kappa^*}\mu|^2 \leq \int_{\Omega'} \kappa^*|\nabla v_e|^2 + \int_{\Omega - \Omega'} \kappa^*|\nabla v_e|^2,$$
where \( v_e = v = \mathcal{H}_\kappa \mu \) in \( \Omega \cap \Omega \) and \( E_\leq \) type extension in \( (\Omega \cap \Omega') \cap \Omega \). Then

\[
\int_{\Omega'} \kappa^*|\nabla v_e|^2 \leq \int_{\Omega' \cap \Omega} \kappa^*|\nabla v_e|^2 = \int_{\Omega' \cap \Omega} \kappa|\nabla v|^2 \leq \int_{\Omega'} \kappa|\nabla v|^2.
\]

Further we note that \( \int_{\Omega - \Omega'} \kappa^*|\nabla v_e|^2 = \int_{\Omega - \Omega'} \kappa^*|\nabla v|^2 \).

The equivalence of energies can be easily used in Lemma 15. We can replace \( k \) by \( k^* \). Then, the condition number of the preconditioned Schur complement \( M^{-1}S \) will not depend on the number of low-conductivity inclusions of \( \kappa \). This partial homogenization will only remove the inclusions and we need additional spectral basis functions, as discussed in Section 3.3 (see (49), to represent disconnected high-conductivity regions that are not isolated within coarse blocks. Partial homogenization will only remove isolated high-conductivity inclusions in this case. One can remove isolated inclusions in a different manner as described in our proceeding paper [13]; however, non-isolated high-conductivity regions (i.e., high-conductivity channels that connect boundaries of the coarse block) need to be represented on a coarse grid. We plan to implement and investigate the advantages of using nonoverlapping domain decomposition methods for the Schur complement in future.

### 6 Conclusions

In this paper, we study domain decomposition preconditioners for multiscale elliptic problems in high-contrast media. We assume that coefficients in each coarse-grid block can have large variations. We construct domain decomposition preconditioners such that the condition number of the preconditioned system is independent of media contrast for general multiscale high-contrast problems. The central part of this work is the construction of coarse spaces that satisfy weighted \( L^2 \) stability requirement with the stability constant independent of the contrast. If a coarse region contains one connected high-conductivity region, then we can use only one basis function to represent the coarse-grid block. For the case with multiple high-conductivity regions, we use weighted...
eigenvalue problem to construct the coarse-scale space. This eigenvalue problem has a number of small asymptotically vanishing eigenvalues. The number of these eigenvalues is the same as the number of connected high-conductivity regions. We show that domain decomposition preconditioners with the coarse space spanned by these eigenvectors have the condition number independent of the contrast. We also study overlapping methods for Schur complement methods. These methods employ harmonic extensions and allow us to use extension theorems to reduce the number of low conductivity inclusions. Numerical results are presented. We compare the proposed methods with domain decomposition methods where multiscale finite element basis or energy minimizing basis functions are used in constructing coarse spaces. We show that the number of iterations is smaller with proposed methods and they remain bounded as the contrast increases.

7 Acknowledgments

This work is partially supported by Award Number KUS-C1-016-04, made by King Abdullah University of Science and Technology (KAUST). YE’s research is partially supported by NSF and DOE. We thanks the anonymous referees for the comments that helped to improve the paper substantially.

A Proof of Lemma 4

Proof. From the usual Poincaré inequality we have

$$\int_{\Omega_0} |v - \overline{v}_0|^2 \leq H^2 \int_{\Omega_0} |\nabla v|^2,$$

where $\overline{v}_0 = \frac{1}{|\Omega_0|} \int_{\Omega_0} v$. This implies

$$\int_{\Omega_0} v^2 \leq H^2 \int_{\Omega_0} |\nabla v|^2 + |\Omega_0| \overline{v}_0^2.$$  \hspace{1cm} (56)

Analogously, for $i = 1, \ldots, M$,

$$\int_{\Omega_i} v^2 \leq H_i^2 \int_{\Omega_i} |\nabla v|^2 + |\Omega_i| \overline{v}_i^2,$$  \hspace{1cm} (57)

where $H_i = diam(\Omega_i)$ and $\overline{v}_i = \frac{1}{|\Omega_i|} \int_{\partial \Omega_i} v$, $i = 1, \ldots, M$. Then

$$\int_{\Omega} \kappa v^2 \leq \sum_{i=0}^{M} \eta_i \int_{\Omega_i} v^2 \leq \sum_{i=0}^{M} \eta_i H_i^2 \int_{\Omega_i} |\nabla v|^2 + \sum_{i=0}^{M} \eta_i |\Omega_i| \overline{v}_i^2.$$  \hspace{1cm} (58)
Now we bound the last sum in (58). We note that
\[ \int_{\Omega} k\nu = \sum_{i=0}^{M} \eta_i |\Omega_i| \nu_i = 0 \quad \text{and then} \quad \eta_0|\Omega_0|\nu_0 = - \sum_{i=1}^{M} \eta_i|\Omega_i|\nu_i. \quad (59) \]

Let \( \Gamma_i = \partial \Omega_i \) and \( c_i = \frac{1}{|\Gamma_i|} \int_{\Gamma_i} v, \ i = 1, \ldots, M. \) Using a Poincaré inequality we obtain
\[ (\nu_0 - \nu_i)^2 \leq \frac{H^2}{|\Omega_0|} \int_{\Omega_0} |\nabla v|^2 + \frac{H^2}{|\Omega_i|} \int_{\Omega_i} |\nabla v|^2 \quad (60) \]
which together with (59) and a discrete Cauchy inequality give
\[ \left( \eta_0|\Omega_0| + \sum_{i=1}^{M} \eta_i|\Omega_i| \right) \int_{\Omega_0} |\nabla v|^2 = \left( \sum_{i=1}^{M} (\nu_0 - \nu_i) \eta_i|\Omega_i| \right)^2 \]
\[ = \left( \sum_{i=1}^{M} (\nu_0 - \nu_i) \sqrt{\eta_i|\Omega_i|} \sqrt{\eta_i|\Omega_i|} \right)^2 \]
\[ \leq \left( \sum_{i=1}^{M} \eta_i|\Omega_i| \right) \left( \sum_{i=1}^{M} (\nu_0 - \nu_i)^2 \eta_i|\Omega_i| \right) \]
\[ \leq \left( \sum_{i=1}^{M} \eta_i|\Omega_i| \right) \sum_{i=1}^{M} \eta_i|\Omega_i| \left( \frac{H^2}{|\Omega_0|} \int_{\Omega_0} |\nabla v|^2 + \frac{H^2}{|\Omega_i|} \int_{\Omega_i} |\nabla v|^2 \right). \]

Then
\[ \left( \beta_0 + \sum_{i=1}^{M} \beta_i \right) \nu_0^2 \leq \left( \sum_{i=1}^{M} \beta_i \right) \frac{H^2}{|\Omega_0|} \int_{\Omega_0} |\nabla v|^2 + \sum_{i=1}^{M} \beta_i \frac{H^2}{|\Omega_i|} \int_{\Omega_i} |\nabla v|^2, \]
where, in order to simplify the notation, we defined \( \beta_i = \eta_i|\Omega_i|, \quad i = 0, 1, \ldots, M. \)

We can bound
\[ \eta_0|\Omega_0|\nu_0^2 \leq \frac{\sum_{i=1}^{M} \beta_i}{\beta_0 + \sum_{i=1}^{M} \beta_i} \eta_0 H^2 \int_{\Omega_0} |\nabla v|^2 \]
\[ + \sum_{i=1}^{M} \left( \frac{\eta_0|\Omega_i|}{\eta_i|\Omega_i|} \beta_i \right) \eta_i H^2 \int_{\Omega_i} |\nabla v|^2 \quad (61) \]
\[ \leq \eta_0 H^2 \int_{\Omega_0} |\nabla v|^2 + \sum_{i=1}^{M} \eta_i H^2 \int_{\Omega_i} |\nabla v|^2, \quad (62) \]
where we have used that \( \frac{\sum_{i=1}^{M} \beta_i}{\beta_0 + \sum_{i=1}^{M} \beta_i} < 1 \) and that \( \frac{\eta_0|\Omega_i|}{\eta_i|\Omega_i|} \beta_i = \frac{\beta_0}{\beta_0 + \sum_{i=1}^{M} \beta_i} < 1. \)

On the other hand,
\[ \eta_j|\Omega_j|\nu_j^2 \leq \eta_j|\Omega_j|\nu_0^2 + \eta_j|\Omega_j|(\nu_j - \nu_0)^2. \quad (63) \]
Using (61) we have

$$\eta_j |\Omega_j| v_0^2 \leq \frac{\sum_{i=1}^M \beta_i}{\beta_0 + \sum_{i=1}^M \beta_i} \eta_j |\Omega_j| v_0^2 \int_{\Omega_0} |\nabla v|^2$$

$$+ \sum_{i=1}^M \frac{\eta_i |\Omega_i| \beta_i}{\beta_0 + \sum_{i=1}^M \beta_i} \eta_i H_i^2 \int_{\Omega_i} |\nabla v|^2 \leq \eta_0 H_0^2 \int_{\Omega_0} |\nabla v|^2 + \sum_{i=1}^M \eta_i H_i^2 \int_{\Omega_i} |\nabla v|^2$$

where we have used the assumption $$\eta_i |\Omega_i| \leq \eta_0, \ i = 1, 2, \ldots, M$$ and

$$\frac{\eta_i |\Omega_i| \beta_i}{\beta_0 + \sum_{i=1}^M \beta_i} = \frac{\beta_j}{\beta + \sum_{i=1}^M \beta_i} < 1.$$ 

Now, using (60) we can bound

$$\eta_j |\Omega_j| (v_j - v_0)^2 \leq \eta_j |\Omega_j| H_j^2 \int_{\Omega_0} |\nabla v|^2 + \eta_j H_j^2 \int_{\Omega_i} |\nabla v|^2$$

$$\leq \eta_0 H_0^2 \int_{\Omega_0} |\nabla v|^2 + \eta_j H_j^2 \int_{\Omega_i} |\nabla v|^2.$$ (64)

With (64) and (64) in (63) we get

$$\eta_j |\Omega_j| v_j^2 \leq \eta_0 H_0^2 \int_{\Omega_0} |\nabla v|^2 + \sum_{i=1}^M \eta_i H_i^2 \int_{\Omega_i} |\nabla v|^2.$$ (65)

Finally, putting (65) and (62) in (58) we get,

$$\int_{\Omega} \kappa v^2 \leq M \eta_0 H_0^2 \int_{\Omega_0} |\nabla v|^2 + M \sum_{i=1}^M \eta_i H_i^2 \int_{\Omega_i} |\nabla v|^2$$

$$\leq M H^2 \int_{\Omega} \kappa |\nabla v|^2.$$ (66)

(B) Auxiliary Proposition

**Proposition 1** For any $$z \in P^3(K)$$ (the space of polynomials of degree 3 or less) we have that

$$\int_{\Omega} \kappa |I^h z|^2 \leq \int_{\Omega} \kappa |z|^2, \quad \int_{\Omega} \kappa |\nabla I^h z|^2 \leq \int_{\Omega} \kappa |\nabla z|^2.$$ (68)

Proof.
Let \( e \subset K \) be an fine scale element. We have
\[
\int_e |I^h z|^2 \leq \sum_{x_k \in e} z(x_k)^2 \int_e |\phi_k|^2 \leq \|z\|_\infty^2 h^2 \leq \|z\|_{L^2(e)}^2 = \int_e z^2,
\]
where we have used an inverse estimate \( \|z\|_\infty \leq h^{-1} \|z\|_{L^2(e)} \) that is valid for all third degree polynomials \( z \) on \( e \); see [6, Lemma 4.5.3]. Multiplying by \( \kappa(x) = \kappa_e \) and summing over all elements in \( e \subset K \) we get the first estimate in (68).

Analogously given any constant \( c \) if we define the second degree polynomial \( \hat{z} = z - c \) we have
\[
\int_e |\nabla I^h \hat{z}|^2 \leq \sum_{x_k \in e} \hat{z}(x_k)^2 \int_e |\nabla \phi_k|^2 \leq \|\hat{z}\|_\infty^2 \leq \|\hat{z}\|_{H^1(e)}^2 \quad (69)
\]
where we have used an inverse estimate \( \|\hat{z}\|_\infty \leq \|\hat{z}\|_{H^1(e)} \) that is valid for all third degree polynomials \( \hat{z} \) on \( e \), Lemma 4.5.3, [6]. Now by choosing \( c \) as the mean value of \( z \) on the element \( e \), we can apply a Poincaré inequality to obtain
\[
\int_e |\nabla I^h z|^2 \leq \|z - c\|_{H^1(e)}^2 \leq \int_e |\nabla z|^2. \quad (70)
\]

Multiplying by \( \kappa_e \) and summing over all elements in \( K \) we get the second estimate in (68).

References


