Chapter 3

Linear Programming: A Geometric Approach

3.1 Graphing Systems of Linear Inequalities in Two Variables

The general form for a line is $ax + by + c = 0$. The general form for a linear inequality is

$$ax + by + c \geq 0 \quad or \quad \begin{array}{l} ax + by + c > 0 \quad or \\ ax + by + c \leq 0 \quad or \\ ax + by + c < 0 \quad . \end{array}$$

How does these inequalities look graphically?

Example - show the inequality $2x - 3y \geq 12$.

Answer - We start with the $=$ part. Graph the line $2x - 3y = 12$.

This line divides the $xy$ plane into an upper half plane and a lower half plane. One side of the line is the solution. Which one? Let’s look at some points in the upper half plane.

Try the point $(0,0)$ in our inequality: $0 - 0 \geq 12$ FALSE.

Try another point in the upper half plane, like $(0,2)$: $0 - 3(2) \geq 12$ FALSE.

In fact all of the points in the upper half plane will not make the inequality true. How about the other half plane?

Try the point $(0,-6)$: $0 - 3(-6) \geq 12$ TRUE. Any of the points in the lower half plane will make the inequality true.

NOTE - you cannot use any points that are ON the line because those points are equal to 12. So if your line passes through the origin, you must take a different point for a test point.

So the solution to this linear inequality is the lower-half plane bounded by the line $2x - 3y = 12$. We really only need one test point from one of the half-planes in determining our solution as if it is not in one half-plane, its in the other. The origin is the easiest test point, but it won’t work if the line goes through the origin. In that case you have to choose another point that is easy to calculate. To show the solution we shade the region that is NOT TRUE. So since the lower half-plane is the solution, we will shade the upper half-plane. See the student calculator manual for more help on shading, especially with the TI-82.

If our inequality had $\geq$ or $\leq$ we draw the bounding line as a solid. If our inequality had $>$ or $<$ we draw the bounding line as DASHED. The region that satisfies our inequality is called the feasible region. This is the region that is white (unshaded).

What if we have two inequalities (a system)? The feasible region will be where they are both true at the same time.

Example: Find the feasible region for the system

\[
\begin{align*}
3x + 2y &> 6 \\
x &\leq 4
\end{align*}
\]

Start by drawing the two lines. (We can’t really draw vertical lines on the calculator, so we want to draw one with a VERY large slope instead. We want the line going through $x = 4$ with a bigggg negative slope. We can use $y = -10^9(x - 4)$) Remember, the first one will be dashed and the second will be solid. Label the intersection point.

The $xy$ plane is divided into 4 regions. Use $(0,0)$ as the test point for each equation. We see that $(0,0)$ does not work in the first equation, so it is the upper half plane that is true. Shade the lower half plane.
Try (0,0) in the second equation. It works and so the left half plane is true so we shade the right half-plane. With a negative slope, the right half-plane will be the upper on shading. You should just check when you shade that the correct region is left clear.

Remember, the shading on the calculator is for your benefit - I will only see what you shade on the paper when you turn it in. Therefore you can just shade by hand on your paper if you wish. I am doing this on the overhead, so I have to use the calculator. The region left clear is the feasible region. It is important to mark the corners of the feasible region. This one has only one corner at (4,-3). It is UNBOUNDED as it continues upward without bound. In general, a region is unbounded if it cannot be enclosed in a circle. If it can, it is called BOUNDED.

Example: Shade the feasible region and label the corner points for the following system of linear inequalities:

\[
\begin{align*}
4x - 3y & \leq 12 \\
x + 2y & \leq 10 \\
x & \geq 0 \\
y & \geq 2
\end{align*}
\]

Start by graphing the system. You do not need to graph the line \(x = 0\), it is the \(y\)-axis. If I want only the region of \(x \geq 0\), I will just start my window at XMIN=0. That will also avoid shading a vertical line!

Use test points to find:

Y1 is true for (0,0) so we shade the lower half
Y2 is true for (0,0) so we shade the upper half
Y3 is false for (0,0) so we shade the lower half

Our feasible region looks like:
We see this is bounded region (if we make the circle large enough we can enclose the entire region). Now to find the coordinates of the 4 corners. We can use the intersection button and the trace feature (for where it crosses the $y$-axis):

Notice that not all the intersections are corners of the feasible region. Also notice that the points are given as decimals when you find the intersections on the calculator but you must give EXACT (fraction) answers for full credit.

### 3.2 Linear Programming Problems

In real life linear inequalities are important because we often want to minimize or maximize a quantity (called the objective function) subject to certain constraints (linear inequalities). For example, we may want to maximize our profits but we are constrained by how much material and labor are available. Or we may want to minimize the calories in our diet subject to getting at least our daily requirements of vitamins. In this section we will only SET-UP the problems. Solving them isn’t until the next section.

Production Scheduling: A company produces two models of hibachis, model A and model B. To produce each model A requires 3 lbs of cast iron and 6 minutes of labor. Each model B requires 4 lbs of cast iron and 3 minutes of labor. The profit for each model A is $2 and the profit for each model B is $1.50. If 1000 lbs of cast iron and 20 hours of labor are available each day, how many hibachis of each model should be produced to maximize profits?

Start by defining your variables! We are asked how many of each model, so define:

\[ x = \text{number of model A produced per day} \]
\[ y = \text{number of model B produced per day}. \]

Next we want to find the objective function. That is the quantity we are trying to minimize or maximize. Here we want to maximize profits, so say

\[ P = 2x + 1.5y \text{ Maximize}. \]
We next want the constraints. We only have a certain amount of iron and labor available each day, so we say

subject to

\[ 3x + 4y \leq 1000 \text{ use 1000 lbs or less of iron} \]

\[ 6x + 3y \leq 20 \times 60 = 1200 \text{ use 1200 minutes or less of labor (watch the units!)}. \]

Finally, this is a word problem so we cannot have a negative number of hibachis,

\[ x \geq 0 \text{ and } y \geq 0 \text{ (non-negativity condition)} \]

Allocation of Funds: A bank has $20 million available for homeowner and auto loans. The homeowner loans average 10% return and the auto loans 12%. The total amount of homeowner loans must be at least four times the amount of auto loans. Determine how much should be used for each type of loan to maximize the bank’s return.

We need to find how much of each type of loan, we will let

\[ x = \text{amount of homeowner loans} \]

\[ y = \text{amount of auto loans} \]

We want to maximize returns, so the return will be our objective function to maximize:

\[ R = .10x + .12y \text{ maximize} \]

subject to

\[ x + y \leq 20,000,000 \text{ available funds} \]

\[ x \geq 4y \text{ ratio} \]

\[ x \geq 0 \text{ and } y \geq 0 \text{ non-negativity.} \]

Check the ratio equation - from the problem it says we want more homeowner loan than auto loans. Test with $1 in auto loans, \( y = 1 \),

\[ x \geq 4y \rightarrow x \geq 4 \cdot 1 \rightarrow x \geq 4 \text{ and that is what we want!} \]
3.3 Graphical Solution of Linear Programming Problems

When our linear programming problem has only two variables we can solve the problem graphically. The constraint equations are linear inequalities that correspond to a feasible region in the $xy$ plane. How does the objective function fit in?

Consider our hibachi problem. We had

$x =$ number of model A hibachis

$y =$ number of model B hibachis

maximize $P = 2x + 1.5y$

subject to

$3x + 4y \leq 1000$ pounds of iron available

$2x + y \leq 400$ minutes of labor available

$x \geq 0$

$y \geq 0$

Make a sketch of the feasible region. Label the feasible region $S$. Any point in $S$ will satisfy the constraints, but which one will maximize our profit?

Look at the point (0,0). The profit there is $P = 2(0) + 1.5(0) = 0$. Where else is the profit zero? There is a line $P = 0 = 2x + 1.5y$ and everywhere along that line the profit is zero. It cuts through the feasible region only in the corner (0,0).

Zero doesn’t seem like much of a profit. How about a profit of $300? There are many ways to make a profit of $300. If we sell 150 of model A (profit of $2 each) we will get $300. If we sell 200 of model B (profit of $1.50 each) we will get $300. So again there is a line $P = 300 = 2x + 1.5y$. This line is parallel to the first line of zero profit (they have the same slope, different intercepts). This passes through much more of the feasible region. And every point of the line inside the feasible region is a possible way to get $300. Notice that the line for a $300 profit has moved up from the zero profit.

Can we make even more money? We can have a series of parallel lines moving through the feasible region with $P$ increasing with each line. How large can $P$ get and still be in $S$? Let’s have some theorems to help

Theorem 1: If a linear programming problem has a solution, then it must occur at a vertex (or corner point) of the feasible region $S$ associated with the problem. Furthermore, if the objective function is optimized at two adjacent vertices of $S$, then it is optimized at every point on the line segment joining these vertices. In this case there are infinitely many solutions.

So the answer is that the profit will be maximized at a corner of the feasible region. If you look at the sequence of parallel lines we
found, there is no way to get out of the feasible region (as we let \( P \) get bigger or smaller) except through a corner point (or two corner points at the same time). However, theorem 1 only says IF there is a solution. Now have:

Theorem 2: Existence of a Solution

Suppose we are given a linear programming problem with a feasible set \( S \) and objective function \( P = ax + by \).

1. If \( S \) is bounded then \( P \) has both a maximum and a minimum value on \( S \).
2. If \( S \) is unbounded and both \( a \) and \( b \) are non-negative, then \( P \) has a minimum value on \( S \) provided that the constraints defining \( S \) include \( x \geq 0 \) and \( y \geq 0 \).
3. If \( S \) is the empty set, then the linear programming problem has no solution. That is, \( P \) does not have a minimum or a maximum.

So we will use the METHOD OF CORNERS in this section to solve the linear programming problems:

1. Graph the feasible region (set), \( S \).
2. Find the coordinates of all the vertices (corner points) of \( S \).
3. Evaluate the objective function at each vertex.
4. Find the vertex that renders the objective function a maximum (or minimum). If there is one such vertex, that is the unique solution to the linear programming problem. If the objective function is maximized (or minimized) at two adjacent vertices of \( S \), there are infinitely many optimal solutions given by the points on the line segment connecting the two vertices.

So in our hibachi example we would make a table (ALWAYS make me a table to show your work):

<table>
<thead>
<tr>
<th>vertex</th>
<th>( P = 2x + 1.5y )</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td>0</td>
<td>minimum</td>
</tr>
<tr>
<td>(0,250)</td>
<td>375</td>
<td></td>
</tr>
<tr>
<td>(120,160)</td>
<td>480</td>
<td>maximum</td>
</tr>
<tr>
<td>(200,0)</td>
<td>400</td>
<td></td>
</tr>
</tbody>
</table>

To maximize profits make 120 of type A hibachis and 160 of type B hibachis.

Example - Shipping parts: A part is shipped each week from a warehouse to assembly plants in Hobbs and Sacramento. The plant in Sacramento needs at least 1000 of these parts each week and the plant in Hobbs needs at least 1200 of these parts each week. It costs $2 to ship each part to Sacramento and $3 to ship each part to Hobbs. The budget limits the shipping costs to $14,000 per week. The work hours required to ship a part to Sacramento are .2 hours and to Hobbs is .25 work hours. How many parts should be shipped to each location if the work hours are to be kept to a minimum?

Answer: Shipping problems are often best visualized with a drawing

\[
\begin{align*}
&\text{warehouse} \\
\text{y} &\text{Sacramento} \\
&\text{Hobbs} \\
&\text{at least 1200} \\
&\text{at least 1000} \\
&.25 \text{ hr.} \\
&.20 \text{ hr.} \\
&\$3 \\
&\$2
\end{align*}
\]

\( x = \text{number of parts shipped to Sacramento} \)

\( y = \text{number of parts shipped to Hobbs} \)

Minimize \( W = .2x + .25y \)

Subject to

\[
\begin{align*}
2x + 3y &\leq 14,000 \quad \text{shipping dollars available} \\
x &\geq 1000 \\
y &\geq 1200
\end{align*}
\]
Sketch the feasible region:

\[ Y_1 = 1000 - 2x \]
\[ Y_2 = 1200 \]
\[ Y_3 = 10^{-8}(x - 1000) \]
\[ Y_4 = - \]
\[ Y_5 = \]

Make a table of the vertex and the value of the objective function:

<table>
<thead>
<tr>
<th>vertex</th>
<th>( W = .2x + .25y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1000,1200)</td>
<td>500</td>
</tr>
<tr>
<td>(1000,4000)</td>
<td>1200</td>
</tr>
<tr>
<td>(5200,1200)</td>
<td>1340</td>
</tr>
</tbody>
</table>

So they should ship 1000 parts to Sacramento and 1200 parts to Hobbs.

Example: A dietitian is to prepare two foods in order to meet certain requirements. Each pound of food I contains 100 units of vitamin C, 40 units of vitamin D and 20 units of vitamin E and costs 20 cents. Each pound of food II contains 10 units of vitamin C, 80 units of vitamin D and 15 units of vitamin E and costs 15 cents. The mixture of the two foods is to contain at least 260 units of vitamin C, 320 units of vitamin D and 120 units of vitamin E. How many pounds of each type of food should be used in order to minimize the cost?

Answer:

\( x = \text{number of pounds of food I} \)
\( y = \text{number of pounds of food II} \)

Minimize \( C = .2x + .15y \)

Subject to

\[
\begin{align*}
100x + 10y & \geq 260 & \text{vitamin C} \\
40x + 80y & \geq 320 & \text{vitamin D} \\
20x + 15y & \geq 120 & \text{vitamin E} \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

Sketch the feasible region. It is unbounded, but we are trying to get a minimum.

Make a table of the vertex and the value of the objective function:

<table>
<thead>
<tr>
<th>vertex</th>
<th>( C = .2x + .15y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,26)</td>
<td>3.9</td>
</tr>
<tr>
<td>(8,0)</td>
<td>1.6</td>
</tr>
<tr>
<td>(4.8,1.6)</td>
<td>1.2</td>
</tr>
</tbody>
</table>
| (27/13.68/13) | 1.2                 | minimum

minimum
The cost is minimized along a line segment connecting the points (4.8, 1.6) and (27/13, 68/13) (≈ (2.1, 5.2)). This is along the line $20x + 15y = 120$ so we can say the solution is

$$y = \frac{120 - 20x}{15} \text{ or } y = -\frac{4}{3}x + 8 \text{ with } 27/13 \leq x \leq 4.8$$

and there are an infinite number of points that minimize the cost.