4.1 The Simplex Method: Standard Maximization Problems

The simplex method is a method of algebraically finding the corner points for a linear programming problem with \( n \) variables. It can be applied to several kinds of linear programming problems; however, we will only do standard maximization problems here. These are problems which meet the following conditions:

1. the objective function is to be maximized
2. all the variables in the problem are non-negative
3. each of the constraints can be written so that the expression containing the variables is \( \leq \) a non-negative constant.

Let us start with a problem with two variables and see how the method of corners and the simplex method are related.

Maximize \( P = 5x + 3y \)

subject to
\[ x + y \leq 80 \]
\[ x \leq 30 \]
\[ x \geq 0 \] and \( y \geq 0 \)

First check that this is a standard maximization problem. YES.

First I will graph the feasible region:

There are four corners: (0,0), (0,80), (30,50) and (30,0). Any of the four could be where \( P \) is maximized.
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The first step in simplex is to change our linear inequalities into linear equalities by introducing SLACK VARIABLES. The first of our linear inequalities would become:

\[ x + y + u = 80 \]

We have introduced the variable \( u \) into the equation. \( u \) will “take up the slack” between the value of \( x + y \) and 80 to make this an equation. For example, look at the point \( x = 10 \) and \( y = 30 \). We find here that \( x + y = 10 + 30 = 40 \leq 80 \). When we have \( u \) in the equation we will have \( 10 + 30 + u = 80 \) and so our slack variable \( u = 40 \). It took up the difference between the value using \( x \) and \( y \) and the number in the inequality.

The line we graphed was \( x + y = 80 \). So everywhere along the line we have \( u = 0 \). In the feasible region we have \( x + y < 80 \) and so \( u \) is a positive number to take up the difference between \( x + y \) and 80. Outside the feasible region we have \( x + y > 80 \) and so there \( u \) is negative to bring the total of \( x \) and \( y \) to be 80. So we can say that our slack variable is positive in the feasible region. Convert the other equation so that we have the system:

\[
\begin{align*}
x + y + u & = 80 \\
3x + v & = 30
\end{align*}
\]

We also have the non-negativity constraint, so inside the feasible region ALL our variables, \( x, y, u \) and \( v \) must be positive. Let us find the value of all the variables at each corner as \((x, y, u, v)\):

\[
\begin{align*}
(0,0,80,30) \\
(30,0,50,0) \\
(30,50,0,0) \\
(0,80,0,30)
\end{align*}
\]

see a pattern? at each vertex two of the variables are zero.

Next step is to rewrite the objective function so there is a 0 on the right hand side (the side we always have the constants). This makes the objective function a variable also:

\[-5x - 3y + P = 0\]

It is important for the simplex method to work that you have the objective function be positive in your equation and 0 on the right hand side. We now have a system of 3 equations and 5 unknowns:

\[
\begin{align*}
x + y + u + \dot{v} & = 80 \\
x + v & = 30 \\
-5x - 3y + P & = 0
\end{align*}
\]
This may seem like it has made things worse, but it really hasn’t. This system has an infinite number of solutions. We can solve for 3 of the variables in terms of the other 2 (which we consider as parameters). But remember, our objective function can only be maximized at a vertex and that is what will limit our solution to a single value.

Put our system into an augmented matrix. This is called the INITIAL SIMPLEX TABLEAU:

\[
\begin{bmatrix}
  x & y & u & v & P \\
  1 & 1 & 1 & 0 & 0 & | & 80 \\
  1 & 0 & 0 & 1 & 0 & | & 30 \\
  -5 & -3 & 0 & 0 & 1 & | & 0 \\
  NB & NB & B & B & B & |
\end{bmatrix}
\]

We can see from this matrix that there are two types of columns. Nice ones that have only 0’s and 1’s (called unit columns) and messy ones that have other numbers in them. Each column corresponds to a different variable. You should label them along the top. The variables with the nice columns (all zeros except for one 1) are called BASIC variables. I will put B under those three columns in this matrix.

The variables that are in the messy columns (messy means not all 0’s and one 1 - sometimes they are not too messy looking) are called NON-BASIC. I will put a NB under those columns. Here we have found that \(x\) and \(y\) are NB and \(u\), \(v\) and \(P\) are B variables. Our two NB variables will be our parameters. We can write out the values of our B variables in terms of our parameters, the two NB variables:

\[
\begin{align*}
\text{from row 1} & \quad u = -x - y + 80 \\
\text{from row 2} & \quad v = -x + 30 \\
\text{from row 3} & \quad P = 5x + 3y
\end{align*}
\]

A particular solution to this system is to let our parameters be zero. In that case we have:

\[u = 80, \quad v = 30\] and \(P = 0\).

Or, to write it as \((x, y, u, v)\) it is \((0, 0, 80, 30)\). But this was one of our corner points on the graph! So when we let our two parameters (NB variables) be zero, we get a corner point.

Have we found a maximum for \(P\) yet? In our original equation for \(P\) we had \(P = 5x + 3y\). That says that if \(x\) or \(y\) increase that \(P\) will get larger. In our augmented matrix we have a \(-5\) in the bottom row of the \(x\) column and a \(-3\) in the bottom row of the \(y\) column. When these numbers are negative down here that means that if \(x\) or \(y\) is increased then the objective function will increase. This is what we want because we are doing standard maximization problems.

So we are at a corner, but we have not found the maximum for \(P\). We want to try to get to the maximum of \(P\) as quickly as possible so we will first let the \(x\) value increase as \(P\) increases by 5 units for every increase of one unit in
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$x$. So let’s leave $y$ as zero in our parametric equations and see how $x$ effects $u$ and $v$:

\[
\begin{align*}
u &= -x - y + 80 \rightarrow -x + 80 \\
v &= -x + 30
\end{align*}
\]

We know that $u$ must be non-negative to be in $S$ and so the first equation tells us that $x \leq 80$.

We know that $v$ must be non-negative to be in $S$ and so the second equation tells us that $x \leq 30$.

Since both $u$ AND $v$ must be non-negative, we must go with the stronger of the two conditions and let $x = 30$. With $x = 30$ and $y = 0$, what are $u$, $v$ and $P$ from our parametric equations?

$x = 30$, $y = 0$, $u = -30 + 80 = 50$, $v = -30 + 30 = 0$, and $P = 5(30) + 3(0) = 150$.

or $(30,0,50,0)$ - another one of the corners of the feasible region. And now we have moved along to get $P$ up to 150.

How can we do this in our matrix? First we must define PIVOTING. When you pivot on an element in a matrix you make that element into a 1 and the rest of the column into 0’s. (We did this in G-J). In otherwords, we want to make $x$ into a unit (B) column

What we did was we worked with $x$, which is our first column in the augmented matrix. We then saw how big $x$ could get, and we were limited by our second equation, the second row of the matrix. So we want to pivot on the first column, second row element, $a_{21}$:

Put the matrix into the calculator. We have some programs to help with the arithmatic of the pivoting. It will not tell you where to pivot, but it will change the pivot element you choose into a 1 and the rest of the column 0:
Write out the matrix again and see where are B and NB variable are:

\[
\begin{bmatrix}
  x & y & u & v & P \\
  0 & 1 & 1 & -1 & 0 & | & 50 \\
  1 & 0 & 0 & 1 & 0 & | & 30 \\
  0 & -3 & 0 & 5 & 1 & | & 150 \\
\end{bmatrix}
\]

We will always let the NB variables be 0 and so when I read across the row to write out the equation I will just leave those variables out. We can only be at the corner when those are zero and we know the solution is only at the corner.

Going across row 1: \( u = 50 \), going across row 2: \( x = 30 \), going across row 3: \( P = 150 \)

So we will move from corner to corner by pivoting. You do not need to write out the equations to decide which element to pivot on. We can do all the work within the augmented matrix.

1. The first decision to make is the pivot column. We want to get to the maximum as quickly as possible. We saw that the bottom row has the objective function and that the more negative the coefficient is in this row, the faster the objective function will get larger. So we want to choose the pivot column with the most negative entry in the bottom row. Mark this row with an arrow.

2. Next we must choose the pivot row. We want to let the pivot variable get as large as possible without ever getting negative. So we want to check the ratio of the variable coefficient to the constant to see how big it can get. Find the quotient \( q = a/b \) for each row where \( a \) is the value in the column of constants and \( b \) is a POSITIVE value of an entry in the pivot column. We want only \( b > 0 \) as we cannot divide by zero and if \( b \) is negative it will throw us out of the feasible region. The pivot row will be the row with the smallest non-negative quotient.

3. The pivot element is the element in the pivot row and pivot column.

Now let us see how our original matrix looked and how we would make our choices:
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\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 | & 80(q = 80/1) \\
1 & 0 & 0 & 1 & 0 | & 30(q = 30/1) \\
-5 & -3 & 0 & 0 & 1 | & 0 \\
\uparrow
\end{bmatrix}
\]

After the pivot we have

\[
\begin{bmatrix}
0 & 1 & 1 & -1 & 0 | & 50(q = 50/1) \\
1 & 0 & 0 & 1 & 0 | & 30(q DNE) \\
0 & -3 & 0 & 5 & 1 | & 150 \\
\uparrow
\end{bmatrix}
\]

We have a negative number in the bottom row, that means that the objective function can still be increased. So we then look at the quotients. Only the top row is possible to look at. so we will pivot on row 1, column 2:

\[
\begin{bmatrix}
x & y & u & v & P |
0 & 1 & 1 & -1 & 0 | & 50 \\
1 & 0 & 0 & 1 & 0 | & 30 \\
0 & 0 & 3 & 2 & 1 | & 300 \\
B & B & NB & NB & B |
\end{bmatrix}
\]

We no longer have negative values in the bottom row. That means there is no way to increase the objective function further. So we are done and can write out the solution by reading across the rows:

row 1: \( y = 50 \) and row 2: \( x = 30 \) and row 3: \( P = 300 \)

So we are again at a corner and we have found the maximum value for \( P \).

This algorithm is called the SIMPLEX METHOD:

1. **Set up the initial tableau**
   - take the system of linear inequalities and add a slack variable to each inequality to make it an equation. Be sure to line up variables to the left of the =’s and constants to the right.
   - Rewrite the objective function in the form
     \[-c_1x_1 - c_2x_2 - \ldots - c_nx_n + P = 0\]
     Note the variables are all on the same side of the equation as the objective function and the objective function gets the positive sign.
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- Put this linear equations into an augmented matrix. This is the initial simplex tableau.

2. Look at the bottom row to the left of the vertical line. If all the entries are NON-NEGATIVE, simplex is done. go to step 4. If there are negative entries, the pivot element must be selected.

3. Select the pivot element

- find the most negative entry in the bottom row to the left of the vertical line. This is the pivot column. Mark it with an arrow.
- find the quotients of the constant column value divided by the pivot column number, \( q = a/b \). Only positive numbers are allowed for \( b \).
- Choose the row with the smallest non-negative quotient. This is the pivot row.
- Perform the pivot
- Go to step 2

4. Solution - write out the equations row by row. Remember, if a column is messy it is a NB variable and has a value of 0.

Example: Maximize \( P = x + 2y - z \). Subject to

\[
\begin{align*}
2x + & y + z \leq 14 \\
4x + & 2y + 3z \leq 28 \\
2x + & 5y + 5z \leq 30 \\
x \geq 0 & \quad y \geq 0 \quad z \geq 0
\end{align*}
\]

Introduce the slack variables. The non-negativity equations do not need slack variables. The simplex method only will work when the variables are non-negative, so that is built into the method.

\[
\begin{align*}
2x + & y + z + u = 14 \\
4x + & 2y + 3z + v = 28 \\
2x + & 5y + 5z + w = 30 \\
-x - & -2y + z + P = 0
\end{align*}
\]

write the INITIAL SIMPLEX TABLEAU:

\[
\begin{bmatrix}
2 & 1 & 1 & 1 & 0 & 0 & 0 & | & 14(q = 14/1 = 14) \\
4 & 2 & 3 & 0 & 1 & 0 & 0 & | & 28(q = 28/2 = 14) \\
2 & 5 & 5 & 0 & 0 & 1 & 0 & | & 30(q = 30/5 = 6) \leftarrow \\
-1 & -2 & 1 & 0 & 0 & 0 & 1 & | & 0 \\
\end{bmatrix}
\]
Find the most negative entry in the bottom row. This is the pivot column. Find the quotient of the pivot column number and the number in the row of constants. The smallest nonnegative quotient gives the pivot row. Perform the pivot on the pivot element. Do this on the calculator to find

\[
\begin{bmatrix}
8/5 & 0 & 0 & 1 & 0 & -1/5 & 0 | & 8(q = 8/(8/5) = 5) \\
16/5 & 0 & 1 & 0 & 1 & -2/5 & 0 | & 16(q = 16/(16/5) = 5) \\
2/5 & 1 & 1 & 0 & 0 & 1/5 & 0 | & 6(q = 6/(2/5) = 15) \\
-1/5 & 0 & 3 & 0 & 0 & 2/5 & 1 | & 12
\end{bmatrix}
\]

There are still negative entries in the bottom row, so we need to chose the pivot column and the pivot row. There is only one negative entry in the bottom row, so we choose column 1. There is a tie for the smallest quotient. When this happens, you may pick any of the lowest valued rows. Here I did row 1, but if I had chosen row 2 I would get the same answer. Perform the pivot:

\[
\begin{bmatrix}
x & y & z & u & v & w & P \\
1 & 0 & 0 & 5/8 & 0 & -1/8 & 0 | & 5 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 | & 0 \\
0 & 1 & 1 & -1/4 & 0 & 1/4 & 0 | & 4 \\
0 & 0 & 3 & 1/8 & 0 & 3/8 & 1 | & 13 \\
B & B & NB & NB & B & NB & B
\end{bmatrix}
\]

There are no negative entries in the bottom row, so we are done with simplex. Now write out the solution. We look for nice (B) and messy (NB) columns. The NB columns are the parameters and we set them equal to zero. Often I will lightly cross those columns out so when I read across the rows I will remember they are zero.

NB: \(z = 0\), \(u = 0\), \(w = 0\)

row 1: \(x = 5\), row 2: \(u = 0\), row 3: \(y = 4\), row 4: \(P = 13\)

we have now found the values of all seven of the variables. The answer in \((x, y, z)\) form for the place where \(P\) is maximized is \((5,4,0)\). The maximum value of \(P\) is 13.

Example: A confectioner has 600 pounds of chocolate, 100 pounds of nuts and 50 pounds of in inventory with which to make three types of candy. Sweet Tooth, Sugar Dandy and Dandy Delight. A box of Sweet Tooth uses 3 pounds of chocolate, 1 pound of nuts and 1 pound of fruit and it sells for $8. A box of Sugar Dandy requires 4 pounds of chocolate, 1/2 pound of nuts and sells for $5. A box of Dandy Delight requires 5 pounds of chocolate, 3/4 pound of nuts 1 pound of fruit and sells for $6. How many boxes of each type of candy can be made from inventory available in order to maximize revenue?
Answer: Start with setting up the variables. Then find the objective function and the constraint inequalities.

\( x = \) number of boxes of Sweet Tooth

\( y = \) number of boxes of Sugar Dandy

\( z = \) number of boxes of Dandy Delight

Maximize \( R = 8x + 5y + 6z \) subject to

\( 3x + 4y + 5z \leq 600 \) chocolate, introduce \( u \)

\( 1x + 1/2y + 3/4z \leq 100 \) nuts, introduce \( v \)

\( 1x + 0y + 1z \leq 50 \) fruit, introduce \( w \).

You do not need to write the slack equations out if you write the linear inequality and the slack variable to go with it. Write the initial simplex tableau:

\[
\begin{bmatrix}
3 & 4 & 5 & 1 & 0 & 0 & 0 & | & 600 (q = 600/3 = 200) \\
1 & 1/2 & 3/4 & 0 & 1 & 0 & 0 & | & 100 (q = 100/1 = 100) \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & | & 50 (q = 50/1 = 50) \\
-8 & -5 & -6 & 0 & 0 & 0 & 1 & | & 0 \\
\end{bmatrix}
\]

Do the pivot on row 3 column 1. Next tableau looks like:

\[
\begin{bmatrix}
0 & 4 & 2 & 1 & 0 & -3 & 0 & | & 450 (q = 450/4 = 112.5) \\
0 & .5 & -.25 & 0 & 1 & -1 & 0 & | & 50 (q = 50/.5 = 100) \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & | & 50 (q D N E) \\
0 & -5 & 2 & 0 & 0 & 8 & 1 & | & 400 \\
\end{bmatrix}
\]

Do the pivot on row 2 column 2:}

\[
\begin{bmatrix}
0 & 0 & 4 & 1 & -8 & 5 & 0 & | & 50 (q = 50/5 = 10) \\
0 & 1 & -.5 & 0 & 2 & -2 & 0 & | & 100 (q D N E) \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & | & 50 (q = 50/1 = 50) \\
0 & 0 & -.5 & 0 & 10 & -2 & 1 & | & 900 \\
\end{bmatrix}
\]
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Pivot on row 1 column 6:

\[
\begin{bmatrix}
  x & y & z & u & v & w & P \\
  0 & 0 & .8 & .2 & -1.6 & 1 & 0 & | & 10 \\
  0 & 1 & 1.1 & .4 & -1.2 & 0 & 0 & | & 120 \\
  1 & 0 & .2 & -2 & 1.6 & 0 & 0 & | & 40 \\
  0 & 0 & 1.1 & .4 & 6.8 & 0 & 1 & | & 920 \\
\end{bmatrix}
\]

No negative numbers in the bottom row so simplex is done. Now find our NB variables with their messy rows and set them equal to zero:

- \( z = 0 \)
- \( u = 0 \)
- \( v = 0 \)

Shade these out and write our equations across each row:

- row 1 gives us \( w = 1 \)
- row 2 gives us \( y = 120 \)
- row 3 gives us \( x = 40 \)
- row 4 gives us \( P = 920 \)

This is a word problem and so it deserves a word answer. We have to remember the meaning of ALL our variables...

Make 50 boxes of Sweet Tooth and 120 boxes of Sugar Dandy. Make no boxes of Dandy Delight for a net revenue of $920. We will use all of our chocolate and all of our nuts in inventory. However we will have 10 pounds of fruit left over.

It is IMPORTANT to explain the slack variables when answering a question. The book goes over this briefly, but I feel it is VERY important.

Some further points ... if you have a zero in the column of constants, you will have a quotient that looks like \( q = 0/b = 0 \). Zero is the smallest nonnegative number that I know of. This means you will pivot on the row that has the zero in the constant column (unless the \( b \leq 0 \), in which case you wouldn’t look at the quotient).

Also ... If there is no usable quotient (because they are all negative or division by zero) and you still have negative numbers in the bottom row, then the problem has NO SOLUTION.