Chapter 5

Mathematics of Finance

5.1 Compound Interest

SIMPLE INTEREST is the interest earned on the original principal only. If \( I \) is the interest earned on a principal of \( P \) dollars at interest rate of \( r \) per year for \( t \) years then

\[
I = Prt
\]

To find the total amount after \( t \) years we have the principal plus the interest,

\[
A = P + I = P + Prt = P(1 + rt)
\]

Example - we have $100 invested in an account paying 5% simple interest. It is left in the account for 10 years. How much is in the account at the end of 10 years?

\[
A = 100(1 + .05 \cdot 10) = 150
\]

There is $150 dollars in the account after 10 years of simple interest.

Usually, when the interest is paid it is added to the principal and then interest is earned on the new principal. This is called COMPOUND INTEREST. Let’s consider the previous example, only when the interest is paid at the end of each year we have a new principal to earn interest on:

end of year 1: \( A = 100(1 + .05 \cdot 1) = 100(1.05) = 105 \), this is our new principal and it earns interest for a year,

end of year 2: \( A = 105(1 + .05 \cdot 1) = [100(1.05)](1.05) = 100(1.05)^2 = 110.25 \)

end of year 3: \( A = 110.25(1 + .05 \cdot 1) = 100(1.05)^3 = 115.76 \)

\vdots

end of year 10: \( A = 100(1.05)^{10} = 162.89 \)

Compare this to the $150 we found with simple interest! Sometimes called the "wonder of compounding". We saw a pattern in our calculation. This will be generally true for any interest rate \( r \) and \( t \) years. We can compound as often as we wish. \( m \) is the number of conversions per year:
yearly is once a year, m=1
semi-annually is twice a year, m=2
quarterly is four times a year, m=4
monthly is 12 times a year, m=12
weekly is 52 times a year, m=52
daily is 365 times a year, m=365

The interest paid during each conversion period will be \( i = \frac{r}{m} \) and the number of conversion periods is \( n = mt \). The accumulated amount after \( n \) periods is \( A \),

\[
A = P(1 + \frac{r}{m})^{mt} = P(1 + i)^n
\]

Example - you have $100 invested at 5% for 10 years and it earns compound interest compounded monthly:

\[
P = 100, t = 10, r = .05, m = 12
\]

\[
A = 100(1 + .05/12)^{12\cdot10} = 164.70
\]

We can find this on the calculator. On the TI-83 above the \( x^{-1} \) key is (in yellow) FINANCE. On the TI-83 Plus the Finance function are accessed with the blue APPS button.

Under FINANCE we find the TVM Solver. This is for the Time Value of Money and all our real work in chapter 5 will be done using this function.

On the calculator we have

N is the number of conversion periods. Here it is 120 as we had 10 years with 12 times per year.

I% is the interest rate - USE the percentage - NOT the decimal

PV is the present value

PMT is payments made (use in the next two sections)

FV is the future value

P/Y and C/Y are the payment (cycles) per year. These are the same number for our applications.

PMT - this is when in the cycle the payment is made. We will always use END.
When solving a finance problem there is always one thing that you don’t know. Set that to zero when you are entering your data. After all the data is entered, you solve for the missing information.

Example - use the TVM to find the amount of money in an account paying 5% compounded monthly on a principal of $100 for 10 years:

$N = 10 \times 12 = 120$ (the interest is compounded 120 times)
$I = 5$ (remember not to use decimals here)
$PV = 100$ (our original principal)
$PMT = 0$ (we are not adding any money to the account)
$FV = 0$ (this is what we want to find, so we have to put a 0 here)
$P/Y = 12$ (compounded 12 times per year)

Enter all this into the calculator. Choose FV. Put the cursor on the value you want to find (here it is FV) and then do SOLVE (ALPHA and then ENTER key).

We find in either case FV=-164.7009498. Don’t worry about the negative signs when they come up. You know where the money is going, you just don’t know how much. Here we have to round to two decimal places and find FV=164.70. (You can also go to MODE and set it for two decimal places while we are doing finance). All the calculations in the chapter (except for simple interest) may be done on the calculator or using the formulas. It is up to you - I just want to see the numbers.

Comparing the amount in the accounts for compounding yearly and monthly we find that the more often we compound at the same interest rate we have more money. We can describe an EFFECTIVE INTEREST RATE that is the simple interest rate that would produce the same amount of accumulated interest in one year.

Example - An account earns 8% interest compounded daily. What is the effective interest rate?

Go to FINANCE and scroll down to Eff( and enter. then you are on the home screen and you put in the arguments which are the nominal rate (not in decimals) and the number of conversion periods and we get back 8.328:

Another function on the TI-83 is dbd(. This gives you the days between dates within the period Jan 1, 1950 and Dec 31, 2049.
5.2 Annuities

Sometimes you know how much money you will need at some future date, $A$, but you don’t know how much to put away NOW, $P$ to end up with that amount. Again, we have a formula,

$$ P = A(1 + i)^{-n} $$

or we can solve for PV using the TVM.

Example - You are planning a vacation to Florida in 2 years. You want to have $2000 for the trip. You have an investment option which pays 10% compounded quarterly. How much do you need to invest now to have the money for your vacation?

N=2*4=8, compounded 8 times
I=10
PV=0 (we will solve for it)
PMT=0
FV = 2000
P/Y = 4

Solve for PV and find 1641.89, so put in $1641.49 now, and in two years you will have $2000.

5.2 Annuities

An annuity is an account to which regular payments are made. There are many kinds of annuities, but we will do certain simple annuities. An annuity that is certain and simple has the following properties:

1. The payments are made at fixed time intervals
2. The periodic payments are of equal size
3. The payments are made at the end of the period
4. The conversion (interest paid) is made at the end of the period.

We will use the calculator to find any values needed. The new variable we will use is PMT, this is the size of the periodic payment to the account.

Example - We put $2000 in an IRA that pays 10%. We put the money in each year. How much money is in the account after 10 years? 20 years? 30 years? 40 years?

Use TVM with the following
N = 10 (then 20, 30 and 40)
I = 10
PV = 0
PMT = 2000
FV = 0 (solve for this)
5.2 Annuities

P/Y = 1

after 10 years, FV = 31874.85. You have paid in $2000 per year for 10 years and that is a total of $20,000. So the interest earned is 31,875-20,000 = 11,875.

after 20 years, FV = 114550.00. Interest earned is 114,550 - 20 × 2000 = 74,550.

after 30 years, FV = 328988.05. Interest earned is 328988 - 30 × 2000 = 268,988.

after 40 years, FV = 885185.11. Interest earned is 885,185 - 40 × 2000 = 805,185.

That is why it is important to put money away when you are young - it will sit there and compound, even if you don’t add to it.

Example - You purchase a car for no money down and $299 a month for 5 years. If the interest rate was 12% on the remaining balance, what was the cash price for the car? How much did you pay in interest?

Use TVM with the following:

N=5*12 = 60 payments

I = 12

PV = 0 (solve for this)

PMT = -299

FV = 0 (car is paid off)

P/Y = 12

So the car was worth about PV = 13440. However, you have paid 60*$299 = $17940 in all. So you paid $4500 in interest.

What if you wanted to pay the car off in four years? How much larger would the payments be? How much would you pay in interest?

N=12*4 = 48 payments

I = 12

PV = 13440

PMT = 0 (solve for this)

FV = 0

P/Y = 12

Solve and find PMT=-353.93 or about $354 per month for a payment. You pay this our 48 times, so 48*354 = 16992, so the amount of interest paid is 16992-13440 = 3552 or a savings of about $1000 in interest. You should always watch out for long periods on interest payments - they really add up.

Say you want to save up for a car and not have to pay any interest, but pay in cash. You want to save up for 3 years and have the $13440 at the end of 3 years. If your account earns 8% interest, how much do you need to save each month to have the money ready?

N=3*12 = 36
5.3 Amortization and Sinking Funds

I = 8
PV = 0 (start with nothing in the account)
PMT = 0 (solve for this)
FV = 13440
P/Y = 12

Solve and find you need to put away about $332 per month. This means you will have put in 36*332 = 11952 and so you will have saved 13440-11952=1488 by letting your money compound.

5.3 Amortization and Sinking Funds

When a debt is paid off (as in the car example) the interest is charged on the remaining principal and so most of the first payments are paying the interest, not the principal.

Example - You take out a mortgage for $100,000 at 7.5% for 30 years. The loan is paid off in equal installments and the interest is charged each month on the unpaid balance each month. How much is each payment? How much is paid in all? How much equity do they have in their home after 1 year? (equity is what you have paid towards the principal). How much after 5 years?

Start by finding the monthly payment:
N=30*12=360
I=7.5
PV=-100000 (we owe this much)
PMT=0 (solve for this)
FV=0 (it is paid off)
P/Y = 12

Payments are about $699.21 month. In all we will pay 360*699.21 = 251,715.60 or 251,15.60-100,000 = 151,715.60 in interest.

To find the equity is more complex than our earlier calculations. We are always paying the interest on the unpaid balance. After 1 year we will have made 12 payments and have 348 payments remaining. We then want to find how much the present value of the house is given we are paying $700 per month for 348 more months and it will be paid off at the end of this time.

N=348 (number of payments remaining)
I = 7.5
PV=0 (solve for this)
PMT = 699.21
FV = 0 (paid off)
P/Y =12
5.3 Amortization and Sinking Funds

Solve and find $PV=-99078.17$ or you still owe $99078.17$. So the equity is $100,000 - 99,078.17 = 921.83$. You have only paid $921.83$ towards the principal and $12 \times 699.21 - 921.83 = 7468.74$ in interest.

After 5 years you will have made $5 \times 12 = 60$ payments, so 300 are remaining.

$N=300$ (remaining payments)

$I=7.5$

$PV=0$ (solve for this)

$PMT=699.21$

$FV=0$ (paid off)

$P/Y = 12$

Solve and find $PV = -94,617.44$, or you still owe $94,617.44$. So the equity is $100,000 - 94,617.44 = 5382.56$. You have now paid the bank $699.21 \times 60 = 41952.60$ and $41952.60 - 5382.56 = 36,570.04$ of which was interest!

We can also find how much of each payment is going to the principal and how much to interest. The payments will all be $699.21$, but the amount towards the principal is very small at the beginning of the loan as the interest is charged on the remaining principal.

The monthly interest rate is $i = r/12 = 7.5\%/12 = .625\% = .00625$. The outstanding principal at the end of the first month is $100,000$. We pay interest on that amount, $I = .00625 \times 100,000 = 625$. The rest of the payment is applied to the principal, $699.21 - 625 = 74.21$. So we have equity $= 74.21$ and outstanding principal of $100,000 - 74.21 = 99,925.79$.

At the end of the second month we pay interest on the outstanding principal, $I = .00625 \times 99,925.79 = 624.54$ and the rest of the payment, $699.21 - 624.54 = 74.67$ goes towards the principal. We now owe $99,925.79 - 74.67 = 99,851.12$ and gave equity of $74.21 + 74.67 = 148.88$.

We can show this information in an AMORTIZATION TABLE

<table>
<thead>
<tr>
<th>end of period</th>
<th>payment</th>
<th>towards interest</th>
<th>towards principal</th>
<th>outstanding principal</th>
<th>equity</th>
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<tr>
<td>0 (360)</td>
<td>100,000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>553.41</td>
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<td>612.06</td>
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<td>13,049.24</td>
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<td>631.61</td>
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<td>0</td>
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</tbody>
</table>

Our loan was at 7.5% for 30 years on $100,000.

What are the savings if the rate is 7.25%?
Calculate the two different payment amounts, $699.21 and $682.18.

Total interest savings are

\[ 360 \times (699.21 - 682.18) = 360 \times 17.03 = 6130. \]

A "point" is 1% of the loan amount (here it is $1000).

Which is better,

- a loan at 7.5% with 0 points
- or a loan at 7.25% with 3 points?

Choose the lower rate since it will save you 6 points over the higher interest loan.

A 15-year mortgage has numbers that are a lot less scary. In a similar situation you could expect to borrow $100,000 at 7.25% for 15 years (the rate is always lower for a shorter mortgage). What are the monthly payments and total interest paid?

\[
N = 12 \times 15 = 180 \\
I = 7.25 \\
PV = -100000 \\
PMT = 0 \text{ (solve for this)} \\
FV = 0 \\
P/Y = 12
\]

Solve and find the payment is about $913 per month ($213 more per month than the 30 year mortgage). Total paid over the course of the loan is 180 \times 913 = 164,340 or a total of $64,340 in interest. That is a savings of 152,000 - 64,340 = 88,000 in interest! In a similar way, you will gain equity much faster with the shorter payment time.

In a sinking fund we make payments with a goal of a certain amount of money available at the end of the period.

Example - a family wants to save for their child’s college tuition. They want to have $30000 available in 18 years. The account pays 9% compounded annually. How much should they save each year to have the money ready?

\[
N = 18 \text{ (number of payments)} \\
I = 9 \\
PV = 0 \text{ (start with nothing in the account)} \\
PMT = 0 \text{ (solve for this)} \\
FV = 30000 \\
P/Y = 1
\]

Solve and find you should put away about $725 per year to reach your goal.