The Theory of Games

DECISION MAKING

Game theory is a mathematical model that provides a systematic way to deal with decisions under circumstances where the alternatives are ambiguous or unclear.

Example - A small card manufacturer needs to decide what theme to use for the holiday cards in the upcoming season. She has two themes to choose from, nature or retro. If the economy is going well she can expect to make a profit of $140,000 on the nature theme and a profit of $60,000 on the retro theme. If the economy is doing poorly, she can expect a profit of $40,000 on the nature theme cards and $80,000 on the retro theme cards.

Which theme should she use? First we must define the problem mathematically. In this case there are two STATES OF NATURE. The states of nature are things that you cannot control. Here the states of nature will be the economy. There are two STRATEGIES. Strategies are things that you can control; namely the choice of themes.

The consequence of each course of action (strategy) under each state of nature is called a PAYOFF. We will organize these payoffs in a matrix, called the PAYOFF MATRIX. We typically put the different strategies on the rows and the states of nature on the columns.

\[
\begin{bmatrix}
\text{good} & \text{poor} \\
\text{Nature} & 140,000 & 40,000 \\
\text{Retro} & 60,000 & 100,000
\end{bmatrix}
\]

What will the manufacturer choose to do if she is an OPTIMIST? An optimist thinks that best outcome will occur and he will choose a strategy that will allow the best to happen. Looking at the payoff matrix, the highest payoff is $140,000 if you choose the nature theme and $80,000 if you choose the Retro theme. The $140,000 payoff is the higher of the two and so the optimist will choose the nature theme and assume that the economy will be good.

What will the manufacturer choose to do if she is a PESSIMIST? A pessimist looks at what the worst is that can happen under each strategy. If she chooses the nature theme then the worst that can happen is to make $40,000. If she chooses the retro theme then the worst that can happen is to gain $60,000. So she will choose to the retro theme and assume that the economy will be bad.

What would the manufacturer do if she reads the Wall Street Journal? If she finds there is a 30% chance of a poor economy she can calculate the expected value of the two strategies.

\[E(\text{Nature}) = 140,000 \times .7 + 40,000 \times .3 = 110,000\]

\[E(\text{Retro}) = 60,000 \times .7 + 100,000 \times .3 = 72,000\]

The expected value for Nature is higher, so choose the nature theme.

What if the probability of a poor economy is 60%?

\[E(\text{Nature}) = 140,000 \times .4 + 40,000 \times .6 = 80,000\]

\[E(\text{Retro}) = 60,000 \times .4 + 100,000 \times .6 = 84,000\]

So she should choose the Retro theme.
STRICTLY DETERMINED GAMES

Game theory is a relatively recent mathematical tool developed to help make decision is business, warfare and social situations. It can also be applied to recreational games as well. We will need some definitions for a game played with two players,

We will call the players \( A \) and \( B \) and the game is called a two-person game. Player \( A \) controls the ROW decisions and player \( B \) controls the COLUMN decisions. What the player chooses is called their strategy. The payoff is determined by the intersection of the row chosen by \( A \) and the column chosen by \( B \). The convention is that a positive number in the payoff matrix means that \( B \) pays \( A \). A negative number means \( A \) pays \( B \).

Consider a payoff matrix:

\[
\begin{bmatrix}
3 & -1 \\
-4 & 7
\end{bmatrix}
\]

Then if \( A \) chooses row 1 and \( B \) chooses column 1, the payoff is 3 and \( B \) owes \( A \) 3 (can be $3, 3 points, 3 whatevers ...).

Then if \( A \) chooses row 1 and \( B \) chooses column 2, the payoff is -1 and \( A \) owes \( B \) 1.

Then if \( A \) chooses row 2 and \( B \) chooses column 1, the payoff is -4 and \( A \) owes \( B \) 4.

Then if \( A \) chooses row 2 and \( B \) chooses column 2, the payoff is 7 and \( B \) owes \( A \) 7.

This kind of game where if one person loses the other wins that amount is called a zero-sum game. Some games (like a model of the stock market) are not zero-sum games.

DOMINATED STRATEGIES

Look at a payoff matrix:

\[
\begin{bmatrix}
2 & 3 & 7 & 0 \\
-1 & 5 & 4 & 1 \\
1 & 0 & 2 & -3
\end{bmatrix}
\]

First look at the matrix from \( A \)'s point of view. \( A \) likes positive numbers as that represents a payoff to him.

If \( A \) chooses row 1 then each number in row 1 is larger than the number in the same column in row 3.

In column 1, row 1 has a payoff of 2 vs. the payoff of 1 in row 3.

In column 2, row 1 has a payoff of 3 vs. the payoff of 0 in row 3.

In column 3, row 1 has a payoff of 7 vs. a payoff of 2 from row 3.

In column 4, row 1 has a payoff of 0 vs. a payoff of -3 from row 3.

So why would player \( A \) ever choose row 3 when row 1 was available? He wouldn’t!

So we may as well remove row 3 since it would never be chosen.

We can say that Row 1 DOMINATES Row 3.
We get a new payoff matrix with the dominated strategy removed:

\[
\begin{bmatrix}
2 & 3 & 7 & 0 \\
-1 & 5 & 4 & 1 \\
\end{bmatrix}
\]

When looking if one row dominates another row, you are looking of every entry in the row is larger than the entry in the corresponding column of another row. There can be more than one dominated strategy. Look at the rows of the new matrix.

In column 1 row 1 is better than row 2, but in column 2 row 2 is better than row 1. So neither dominates.

Look at the columns now which is player B’s view. B likes negative numbers, or at least small numbers.

For B, column 4 is better than column 3.

In row 1, column 4 has a payoff of 0 vs. the payoff of -7 in column 3.

In row 2, column 4 has a payoff of -1 vs. the payoff of -4 in column 3.

So column 4 dominates column 3 and we can remove column 3.

\[
\begin{bmatrix}
2 & 3 & 0 \\
-1 & 5 & 1 \\
\end{bmatrix}
\]

Now column 3 is better than column 2 so we can say column 3 dominates column 2 and remove column 2,

\[
\begin{bmatrix}
2 & 0 \\
-1 & 1 \\
\end{bmatrix}
\]

Check columns and neither is better. Check rows again and neither is better, so we are done.

Example: Remove any dominated strategies from the payoff matrix,

\[
\begin{bmatrix}
1 & 4 \\
0 & -1 \\
3 & 5 \\
-4 & 0 \\
\end{bmatrix}
\]

Answer: Compare column 1 and column 2 - neither dominates. Compare row 3 to row 4 and find that row 3 dominates row 4, so remove row 4. New matrix,

\[
\begin{bmatrix}
1 & 4 \\
0 & -1 \\
3 & 5 \\
\end{bmatrix}
\]

Compare row 3 to row 2 - find row 3 dominates row 2 and so remove row 2.

\[
\begin{bmatrix}
1 & 4 \\
3 & 5 \\
\end{bmatrix}
\]
Compare row 1 and row 2 and find row 2 dominates row 1. Remove row 1,

\[
\begin{bmatrix}
3 \\
5
\end{bmatrix}
\]

No more rows to compare. Check the columns again and find column 1 dominates column 2 and remove column 2,

\[
\begin{bmatrix}
3
\end{bmatrix}
\]

The goal of game theory is to find the OPTIMUM STRATEGY for each player. The optimum strategy will give the player the most payoff possible. The payoff that results from each player choosing their optimum strategy is called the VALUE of the game.

The simplest strategy is for a player to always pick the same row (or same column) every time they play. This is called a PURE STRATEGY. Consider the following payoff matrix:

\[
\begin{bmatrix}
1 & -4 & -6 \\
-1 & -2 & 1 \\
2 & -5 & -4
\end{bmatrix}
\]

Now look at each of A’s strategies and see what B would choose as a response to each.

<table>
<thead>
<tr>
<th>A chooses row</th>
<th>B then chooses column</th>
<th>payoff is</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>6 to B</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2 to B</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>5 to B</td>
</tr>
</tbody>
</table>

Based on this, A’s best pure strategy is to choose row 2. We can do this analysis on the payoff matrix itself by underlining the smallest in each row and then finding the row that has the largest underlined value:

\[
\begin{bmatrix}
1 & -4 & -6 \\
-1 & -2 & 1 \\
2 & -5 & -4
\end{bmatrix}
\]

Look at B’s choices for columns and what A would do in response to each:

<table>
<thead>
<tr>
<th>B chooses column</th>
<th>A then chooses row</th>
<th>payoff is</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>2 to A</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2 to B</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1 to A</td>
</tr>
</tbody>
</table>

Here it looks like B should choose column 2. We can show these choices on the payoff matrix by circling (or boxing) the largest number in each column. Then the column that has the smallest number circled shows B’s optimum pure strategy.

\[
\begin{bmatrix}
1 & -4 & -6 \\
-1 & \text{boxed} & -2 & \text{boxed} \\
2 & -5 & \text{boxed} & 1
\end{bmatrix}
\]
In this game we find that the element from row 2, column 2 is the smallest in its row and largest in its column. It is the maximum as far as what $A$ can get from $B$ and the minimum as far as what $B$ has to pay $A$. This is called a SADDLE POINT.

If a game has a saddle point, then the optimum pure strategy for $A$ is to choose the row with the saddle point and the optimum pure strategy for $B$ is to choose the column with the saddle point. A game with a saddle point is a STRICTLY DETERMINED GAME. The value of the game is the payoff at the saddle point. If a game is FAIR then the payoff at the saddle point is zero. The game above has a value of -2 and so it is not fair - it favors $B$.

Example - The BG company and the LP company are bringing out new calculators this fall. The BG company can market this new calculator to the university market or the high school market and the LP company can market this new calculator to the university market or the high school market or both. The profit of each company will depend on how their competitor markets their calculator as shown in the payoff matrix. Let BG control the columns and LP control the rows.

$$
\begin{bmatrix}
U & HS \\
U & 7 & 10 \\
HS & 8 & 2 \\
B & 9 & 11
\end{bmatrix}
$$

Check for a saddle point by underlining the smallest in each row and the largest in each column.

$$
\begin{bmatrix}
U & HS \\
U & 7 & 10 \\
HS & 8 & 2 \\
B & 9 & 11
\end{bmatrix}
$$

Find that the strategy in row 3, column 1 is both circled and underlined. This means that BG should market to the universities and LP should market the calculator to both. Remember, we are finding the best strategy based on the available options, not the maximum possible payoff.

MIXED STRATEGIES

What if a game does not have a saddle point? There is no optimum pure strategy, but a player can have a mixed strategy. Suppose the payoff matrix looks like

$$
\begin{bmatrix}
2 & -1 \\
-4 & 3
\end{bmatrix}
$$

No element is both circled and underlined. That means that no row is best and no column is best all the time. Suppose that player $A$ decides to choose row 1 10% of the time and row 2 90% of the time. That is $P(R1) = .1$ and $P(R2) = .9$.

Next suppose that player $B$ decides to choose column 1 25% of the time and column 2 75% of the time. That is $P(C1) = .25$ and $P(C2) = .75$.

If we assume that $A$ and $B$ make their choices independently then we can multiply to find the probability that a certain payoff is chosen,
<table>
<thead>
<tr>
<th>Outcome</th>
<th>Prob</th>
<th>Payoff to A</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1,C1</td>
<td>.1*.25 = .025</td>
<td>2</td>
</tr>
<tr>
<td>R1,C2</td>
<td>.1*.75 = .075</td>
<td>-1</td>
</tr>
<tr>
<td>R2,C1</td>
<td>.9*.25 = .225</td>
<td>-4</td>
</tr>
<tr>
<td>R2,C2</td>
<td>.9*.75 = .675</td>
<td>3</td>
</tr>
</tbody>
</table>

So the expected value for A is

\[ E = 2 \cdot .025 + -1 \cdot .075 + -4 \cdot .225 + 3 \cdot .675 = 1.1 \]

So, in the long run if these players use those mixed strategies then A will end up ahead about 1.10. If A or B change their strategies, then the expected value of the game will change.

We can find the expected value of a generalized 2 × 2 game as shown:

\[
\text{payoff matrix } = M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}
\]

Let \( p_1 = P(R1) \) and \( p_2 = P(R2) \), then we will write A’s strategy as

\[ A = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \]

Let \( q_1 = P(C1) \) and \( q_2 = P(C2) \) and then we will write B’s strategy as

\[ B = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \]

The expected value is

\[ E = (p_1q_1)a_{11} + (p_1q_2)a_{12} + (p_2q_1)a_{21} + (p_2q_2)a_{22} = p_1q_1a_{11} + p_1a_{12}q_2 + p_2a_{21}q_1 + p_2a_{22}q_2 \]

\[ E = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = AMB \]

So for this game with the strategy given we have

\[ \begin{bmatrix} .1 & .9 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 25 \\ 75 \end{bmatrix} = 1.1 \]

A’s optimum mixed strategy is

\[ p_1 = \frac{a_{22} - a_{21}}{a_{11} - a_{21} - a_{12} + a_{22}} \]

\[ A = \begin{bmatrix} p_1 \\ 1 - p_1 \end{bmatrix} \]

and then B’s optimum mixed strategy is

\[ q_1 = \frac{a_{22} - a_{12}}{a_{11} - a_{21} - a_{12} + a_{22}} \]

\[ B = \begin{bmatrix} q_1 \\ 1 - q_1 \end{bmatrix} \]
The expected value of the game can also be found using matrix multiplication as

\[ E = AMB = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} - a_{21} - a_{12} + a_{22}} \]

We will not look at games with mixed strategies with more than two rows and two columns!

So how should player A and B choose their strategy for the game above?

\[ p_1 = \frac{4 - (-4)}{2 - (-4) - (-1) + 3} = \frac{7}{10} = .7 \quad p_2 = 1 - .7 = .3 \]

\[ q_1 = \frac{3 - (-1)}{2 - (-4) - (-1) + 3} = \frac{4}{10} = .4 \quad q_2 = 1 - .4 = .6 \]

\[ E = \frac{2 \cdot 3 - (-1)(-4)}{2 - (-4) - (-1) + 3} = \frac{2}{10} = .2 \]

This is not a fair game - it favors A.

CATS - When two cats, Euclid and Jamie, play together, their game involves facing each other while several feet apart; each cat must then decide whether to pounce or to "freeze" (to stay motionless until the other cat pounces). Euclid weighs 2 pounds more than Jamie, so if they both pounce, Jamie is squashed and Euclid gains 3 points. If they both freeze, Euclid remains in control of the area and gains 2 points. If Euclid pounces while Jamie freezes, Jamie can put up a good defence, so Euclid gains only 1 point. Euclid is poor at defence, so if he freezes while Jamie pounces, he loses 2 points. Find the payoff matrix for this game. What is the optimum strategy for each cat? What is the value of this game?

Let Jamie control the rows and Euclid control the columns. The payoff matrix is then

\[
\begin{bmatrix}
F & P \\
F & \begin{bmatrix}
-2 & -1 \\
2 & -3
\end{bmatrix}
\end{bmatrix}
\]

\[ p_1 = \frac{-3 - 2}{-2 - 2 - (-1) + (-3)} = \frac{-5}{-6} = \frac{5}{6} \quad p_2 = \frac{1}{6} \]

\[ q_1 = \frac{-3 - (-1)}{-2 - 2 - (-1) + (-3)} = \frac{-2}{-6} = \frac{1}{3} \quad q_2 = \frac{2}{3} \]

\[ E = \frac{(-2)(-3) - (-1)(2)}{-2 - 2 - (-1) + (-3)} = \frac{6 + 2}{-6} = -4/3 \]

So Jamie should freeze 5/6 of the time and pounce 1/6 of the time. Euclid should freeze 1/3 of the time and pounce 2/3 of the time. Overall Euclid can expect to be ahead 4/3. This is not a fair game - it favors Euclid.