Vector Functions (Section 1.3)

A curve of the type \( x = x(t), \ y = y(t) \) is called a parametric curve and the variable \( t \) is the parameter.

**EXAMPLE 1**
Graph the parametric function \( x = t^3 - 2t, \ y = t^2 - t \)

**EXAMPLE 2**
Sketch the curve represented by the parametric equations and then eliminate the parameter to find the Cartesian equation of the curve.
(a) \( x = 2t - 1, \ y = 2 - t, \ -3 \leq t \leq 3 \)

(b) \( x = 2t - 1, \ y = t^2 - 1 \)

\[
x = 2t - 1 \Rightarrow t = \frac{x+1}{2} \\
y = (\frac{x+1}{2})^2 - 1 \Rightarrow y + 1 = \frac{1}{4}(x+1)^2
\]
(c) \( x = \sin^2 \theta, \quad y = \cos^2 \theta \)  

For each value of the parameter \( t \) we may view the point \((x(t), y(t))\) on a parametric curve as the endpoint of a vector \( r(t) = (x(t), y(t)) = x(t)i + y(t)j \)  

**EXAMPLE 3**  
Describe the motion of a particle with position \((x, y)\) or \( r(t) \) as \( t \) varies in the given interval.  

1. \( r(t) = (8t - 3)i + (2 - t)j, \quad 0 \leq t \leq 1 \)  
\[ x(t) = 8t - 3, \quad y(t) = 2 - t \implies t = 2 - y \]  
\[ x = 8(2 - y) - 3 = 16 - 8y - 3 = 8y + x = 13 \]  
\[ \text{a line} \]  
\[ t = 0, \quad x = -3, y = 2 \]  
\[ t = 1, \quad x = 5, y = 1 \]  
\[ \text{particle moves along the line } 8y + x = 13 \text{ from } (-3, 2) \text{ to } (5, 1) \]  

2. \( r(t) = (2 \sin t, 3 \cos t), \quad 0 \leq t \leq 2\pi \)  
\[ x = 2 \sin t \implies \sin^2 t = (x/2)^2 \]  
\[ y = 3 \cos t \implies \cos^2 t = (y/3)^2 \]  
\[ \sin^2 t + \cos^2 t = 1 = (x/2)^2 + (y/3)^2 \]  
\[ = \text{ellipse centered at } (0, 0) \]  
\[ @ t = 0, \quad (0, 3); \quad @ t = \pi/2, \quad (2, 0) \]  
\[ @ t = \pi, \quad (0, -3); \quad @ t = 3\pi/2, \quad (-2, 0) \]  
\[ \text{particle moves along an ellipse starting at } (0, 3) \]  

Consider a line \( L \) as shown. Can we write this as a vector \( r(t) \)?  

The vector equation of a line is given by \( r(t) = r_0 + tv \) where \( r_0 \) is a position vector to a point on the line, \( v \) is a vector parallel to the line, and \( t \) is a scalar.  

**EXAMPLE 4**  
Given the points (-3, 4) and (2, 8), find a vector equation and a parametric equations for the line that passes through these two points.  
\[ \text{a line determined by a point } P_0 \text{ and a direction (slope) } \frac{\overline{v}}{0} \]  
If \( v = \overline{v} \) to \( \overline{a} \), then \( \frac{\overline{v}}{0} = \frac{\overline{a}}{0} \)  
\[ \overline{v} = \begin{pmatrix} 2 \ 8 \end{pmatrix} - \begin{pmatrix} -3 \ 4 \end{pmatrix} = \begin{pmatrix} 5 \ 4 \end{pmatrix} \]  
\[ r_0 = \begin{pmatrix} 2 \ 8 \end{pmatrix} \]  
\[ r = \begin{pmatrix} 2 \ 8 \end{pmatrix} + t \begin{pmatrix} 5 \ 4 \end{pmatrix} \]  
\[ = \begin{pmatrix} 2 + 5t \ 8 + 4t \end{pmatrix} \]  
\[ \implies x(t) = 2 + 5t, \quad y(t) = 8 + 4t \]
EXAMPLE 5  #30
Given the point $P(2,5)$ and vector $a = \langle 3,0 \rangle$, find
(a) a vector equation
(b) parametric equations
(c) a Cartesian equation for a line that passes through the point $P$ and is parallel to $a$.

(a) $\vec{r}(t) = \vec{r}_0 + t \vec{V} = \langle 2,5 \rangle + t \langle 3,0 \rangle = \langle 2+3t,5 \rangle$

(b) $x(t) = 2+3t$, $y(t) = 5$

(c) $y = 5$ (horizontal line)

EXAMPLE 6  #32
Determine if the lines below are parallel, perpendicular or neither.
If the lines are not parallel, find the point of intersection
$L_1: \vec{r}_1(t) = \langle -4+2t,5+t \rangle = \langle -4+2t,5 \rangle + t \langle 2,1 \rangle$
$L_2: \vec{r}_2(t) = \langle 2+3t,4-6t \rangle = \langle 2,4 \rangle + t \langle 3,-6 \rangle$

$\vec{V}_1 \cdot \vec{V}_2 = \langle 2,1 \rangle \cdot \langle 3,-6 \rangle = (2)(3) + (1)(-6) = 0 \Rightarrow \perp$ (orthogonal)

Intersect when $x_1(s) = x_2(t)$
$y_1(s) = y_2(t)$

$-4+2s = 2+3t \Rightarrow (2s-3t = 6) \Rightarrow t = \frac{2s-6}{3}$
$5+s = 4-6t \Rightarrow s+6t = -1 \Rightarrow s = -1 - 6t$

Set $t = 0$ and solve for $s$: $-1 - 6t = s \Rightarrow s = -1 - 6(0) = -1$

$x_1 = 2 = x_2\left(\frac{1}{3}\right) = 2 \Rightarrow s = \frac{1}{3}$

$y_1 = 5 = y_2\left(\frac{1}{3}\right) = 5 \Rightarrow s = \frac{1}{3}$

Intersect at $\langle 2/3,5/3 \rangle$

EXAMPLE 7  #310
An object is moving in the $xy$-plane and its position after $t$ seconds is $\vec{r}(t) = \langle t-3,t^2-2t \rangle$

(a) Find the position of the object at time $t = 5$.
(b) At what time does the object pass through the point $(1, 8)$?
(c) Does the object pass through the point $(3, 20)$?
(d) Find an equation in $x$ and $y$ whose graph is the path of the object.

(a) $\vec{r}(5) = \langle 5-3,5^2-2(5) \rangle = \langle 2,15 \rangle \Rightarrow (2,15)$

(b) $t-3 = 1 \Rightarrow t = 4 \text{ (check } y = 4^2 - 2(4) = 8)$

(c) $t-3 = 3 \Rightarrow t = 6$

$y = (6)^2 - 2(6) = 24 \neq 20 \Rightarrow \text{ No}$

(d) $x = t-3 \Rightarrow x = t + 3$

$y = (x+3)^2 - 2(x+3) = x^2 + 6x + 9 - 2x - 6 = x^2 + 4x + 3$

Since $t > 0$

$x = t-3 \Rightarrow x = -3$