Test II – Key

Instructions: Show all work in your bluebook. Calculators that do linear algebra or calculus are not allowed.

1. Define each space listed and describe the operations of vector addition (+) and scalar multiplication (⋅) corresponding to it.

(a) **(5 pts.)** \(\mathcal{P}_n\) is the set of all polynomials of degree \(n\) or less; that is, \(\mathcal{P}_n = \{a_0 + a_1 x + \cdots + a_n x^n\}\). Here are the operations. If \(p, q \in \mathcal{P}\), \(p(x) = a_0 + a_1 x + \cdots + a_n x^n\) \(q(x) = b_0 + a_1 x + \cdots + b_n x^n\), then
\[
(p + q)(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n.
\]
If \(c\) is a scalar, then \(c \cdot p\) is the polynomial \((c \cdot p)(x) = ac_0 + ca_1 x + \cdots + ca_n x^n\).

(b) **(5 pts.)** \(C^{(1)}[0,1]\) is the set of all functions \(f\) defined and continuously differentiable on the interval \([0,1]\). If \(f, g \in C^{(1)}[0,1]\), then \(f + g\) is defined by \((f + g)(x) = f(x) + g(x)\) and \(c \cdot f\) is defined by \((c \cdot f)(x) = cf(x)\).

2. **(15 pts.)** Determine whether or not the set \(S\) of \(2 \times 2\) matrices \(M = \begin{pmatrix} x & y \\ z & w \end{pmatrix}\) such that \(x + w = 0\) is a subspace of \(M_{2,2}\).

**Solution.** Is \(0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\) is in \(S\)? Yes, since \(x + w = 0 + 0 = 0\). Is \(S\) closed under addition? Let \(M_1 = \begin{pmatrix} x_1 & y_1 \\ z_1 & w_1 \end{pmatrix}\) and \(M_2 = \begin{pmatrix} x_2 & y_2 \\ z_2 & w_2 \end{pmatrix}\) be in \(S\). We need to check whether \((x_1 + x_2) + (w_1 + w_2) = 0\). Rearranging terms, we get \((x_1 + x_2) + (w_1 + w_2) = (x_1 + w_1) + (x_2 + w_2) = 0 + 0 = 0\). Thus, \(M_1 + M_2\) is in \(S\). Is \(S\) closed under scalar multiplication? To check this, we to see whether \(cx + cw = 0\). Again, this is true because \(cx + cw = c(x + w) = c \cdot 0 = 0\)

3. **(15 pts.)** Determine whether or not the set \(\{1, e^x, e^{2x}\}\) is linearly independent in \(C(-\infty, \infty)\).
Solution. Start with the equation $c_1 + c_2 e^x + c_3 e^{2x} \equiv 0$. Differentiate this twice to get $c_2 e^x + 2c_3 e^{2x} \equiv 0$ and $c_2 e^x + 4c_3 e^{2x} \equiv 0$. Set $x = 0$ in the three equations. This results in the system

$$c_1 + c_2 + c_3 = 0, \quad c_2 + 2c_3 = 0, \quad c_2 + 4c_3 = 0$$

Subtracting the second equation from the third gives $2c_3 = 0$, so $c_3 = 0$. Using this in the second equation gives $c_2 + 2 \cdot 0 = 0$, so $c_2 = 0$. Using both values in the first equation then gives $c_1 = 0$. It follows that the set is linearly independent.

4. (10 pts.) Consider $G : C(-\infty, \infty) \to C(-\infty, \infty)$ given by $Gu(x) = \int_0^x e^t u(t)dt$. Show that $G$ is linear and that it is one-to-one.

Solution. By inspection, the domain and range of $G$ are vector spaces. Also, by rules from algebra and calculus, we have:

$$G[u + v](x) = \int_0^x e^t (u(t) + v(t))dt$$

$$= \int_0^x e^t u(t)dt + \int_0^x e^t v(t)dt$$

$$= Gu(x) + Gv(x).$$

Thus $G$ is additive. In addition, if $c$ is a scalar, then we have:

$$G[cu](x) = \int_0^x e^t (cu(t))dt$$

$$= c \int_0^x e^t u(t)dt$$

$$= cG(u)(x).$$

Thus, $G$ is also homogeneous. $G$ thus satisfies the conditions for it to be linear. To see that $G$ is one-to-one, we need to solve for $u$ when $Gu(x) \equiv 0$. The fundamental theorem of calculus implies

$$\frac{d}{dx} \left( \int_0^x e^t (cu(t))dt \right) = e^x u(x) \equiv 0$$

Dividing by $e^x$ then gives us that $u \equiv 0$. This is equivalent to a linear function being one-to-one, so $G$ is one-to-one.
5. **(20 pts.)** Find bases for the column space, null space, and row space of $C$, and state the rank and nullity of $C$. What should these sum to? Do they?

\[ C = \begin{pmatrix} 1 & -3 & -1 & -3 \\ -1 & 3 & 2 & 4 \\ 2 & -6 & 4 & 0 \end{pmatrix} \]

**Solution.** Use row operations to put $C$ in reduced row echelon form.

\[ C \leftrightarrow R = \begin{pmatrix} 1 & -3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

By the various methods described in class, the bases for the column space, row space, and null space are, respectively, follows.

\[ \begin{align*} \{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} \} & , \quad \{ \begin{pmatrix} 1 & -3 & 0 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \} , \\
\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \} & . \end{align*} \]

The rank and nullity of $C$ are both 2. Their sum should be 4, which is the number of columns, and it is.

6. Given that $L : \mathcal{P}_2 \rightarrow \mathcal{P}_2$ be defined by $L(p) = x^2p'' - 2(x - 1)p' + 3p$ is a linear transformation, do the following:

(a) **(10 pts.)** Find the matrix of $L$ relative to the basis $B = \{1, x, x^2\}$.

**Solution.** First we apply $L$ to the basis. $L[1] = 3$, $L[x] = x + 2$, and $L[x^2] = 2x^2 - 4x^2 + 4x + 3x^2 = x^2 + 4x$. The matrix for $L$ then has as columns the coordinate vectors for each of these, and they are in the same order as $B$; hence, the matrix is

\[ A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \]
(b) (5 pts.) Find \([2 - x + x^2]_B\) and use the matrix from part 6a to solve \(L(p) = 2 - x + x^2\) for \(p\).

**Solution.** First, we have

\[
[2 - x + x^2]_B = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}
\]

The differential equation is completely equivalent to the matrix equation \(A[p]_B = [2 - x + x^2]_B\). Let’s put this in augmented form and row reduce it.

\[
\begin{pmatrix} 3 & 2 & 0 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \iff \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{pmatrix}
\]

Hence, \([p]_E = (4 - 5 \, 1)^T\), and so \(p(x) = 4 - 5x + x^2\).

7. Let \(A = \begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix}\).

(a) (10 pts.) Find the eigenvalues and eigenvectors of \(A\).

**Solution.** The characteristic polynomial is

\[
p_A(\lambda) = \det \begin{pmatrix} 2 - \lambda & -1 \\ 2 & 5 - \lambda \end{pmatrix} = \lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4),
\]

and so there two eigenvalues, \(\lambda = 3\) and \(\lambda = 4\). The two systems we need to solve to get the eigenvectors are just

\[
\begin{pmatrix} -1 & -1 & 0 \\ 2 & 2 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -2 & -1 & 0 \\ 2 & 2 & 0 \end{pmatrix}
\]

These give the eigenvectors \((-1 \, 1)^T\) and \((-1 \, 2)^T\), for 3, 4, respectively.

(b) (5 pts.) Use the answer to part 7a to solve \(\frac{dx}{dt} = Ax\).

**Solution** \(x = c_1e^{3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2e^{4t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}\)