Test I

Instructions: Submit your work as a single MSWord document. This is an open-book test, so you may use the text, other books, etc. Please cite the references you use. You may not consult with each other or any person other me. Please email the completed test to me by Wednesday, March 10, 5 pm.

1. Consider the function \( f(x) := 1 - x \) defined on \( 0 \leq x \leq 1 \). (Note: here \( a = 1, \text{not } \pi \).)

   (a) (15 pts.) Find the Fourier sine series for \( f \), and sketch three periods (period = 2) of the function to which it converges pointwise.

   (b) (5 pts.) Sketch three periods (period = 2) of the function to which the Fourier cosine series converges pointwise. (Do not compute the coefficients in the series.)

   (c) (10 pts.) Is either series uniformly convergent? If so, which? Why? Will either series exhibit the Gibbs’ phenomenon? Briefly explain.

2. Let \( g(x) = e^{i\beta x} \) on \(-\pi < x < \pi\), where \( \beta \) is a real number that is not an integer.

   (a) (10 pts.) Show that the complex form of the Fourier series for \( g \) is

   \[
   \frac{\sin(\pi \beta)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\beta - n} e^{inx}.
   \]

   (b) (10 pts.) Use Parseval’s Theorem, the series above, and the fact that \( |e^{i\beta x}| = 1 \) to show that

   \[
   \csc^2(\pi \beta) = \frac{1}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(\beta - n)^2}.
   \]

3. (15 pts.) Use Parseval’s Theorem for Fourier series to formulate and prove a similar theorem for Fourier sine series on \([0, \pi]\), including a definition and condition for convergence in the mean.
4. **(10 pts.)** Use the properties of the Fourier transform given along with this test to show that for any function \( u(x) \) we have

\[
\mathcal{F} \left[ \frac{du}{dx} + xu \right] (\lambda) = i\lambda \hat{u} + i \frac{d\hat{u}}{d\lambda}.
\]

Now, let \( u(x) = e^{-x^2/2} \). Use the result above plus elementary methods for solving 1st order linear ODEs to show that \( \hat{u}(\lambda) = C e^{-\lambda^2/2} \), where \( C \) is a constant. (Find \( C \) for an extra 5 points.)

5. Let \( a > 0 \). Consider the rectangular pulse \( h(x) = \begin{cases} 
1 & -a < x < a \\
1/2 & x = \pm a \\
0 & |x| > a
\end{cases} \).

(a) **(10 pts.)** Find \( \hat{h}(\lambda) \).
(b) **(10 pts.)** Find the convolution \( h * h \).
(c) **(5 pts.)** Find \( \mathcal{F}[h * h](\lambda) \).
Fourier Transform Properties

1. \( \hat{f}(\lambda) = \mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\lambda}dx. \)

2. \( f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\lambda)e^{ix\lambda}d\lambda. \)

3. \( \mathcal{F}[x^n f(x)](\lambda) = i^n \hat{f}^{(n)}(\lambda). \)

4. \( \mathcal{F}[f^{(n)}(x)](\lambda) = (i\lambda)^n \hat{f}(\lambda). \)

5. \( \mathcal{F}[f(x-a)](\lambda) = e^{-i\lambda a} \hat{f}(\lambda). \)

6. \( \mathcal{F}[f(bx)](\lambda) = \frac{1}{b} \hat{f}\left(\frac{\lambda}{b}\right). \)

7. \( \mathcal{F}[f \ast g] = \sqrt{2\pi} \hat{f}(\lambda)\hat{g}(\lambda) \)

Integrals

1. \( \int udv = uv - \int vdu \)

2. \( \int \frac{dt}{t} = \ln |t| + C \)

3. \( \int e^{at}dt = \frac{1}{a} e^{at} + C \)

4. \( \int t^n e^{at}dt = \frac{1}{a} t^n e^{at} - \frac{n}{a} \int t^{n-1}e^{at}dt \)

5. \( \int e^{at}\cos(bt)dt = \frac{e^{at}}{a^2+b^2} (a \cos(bt) + b \sin(bt)) + C \)

6. \( \int e^{at}\sin(bt)dt = \frac{e^{at}}{a^2+b^2} (a \sin(bt) - b \cos(bt)) + C \)

7. \( \int t \sin(t)dt = \sin(t) - t \cos(t) + C \)

8. \( \int t \cos(t)dt = \cos(t) + t \sin(t) + C \)