Homework assignment #5
(due Thursday, October 5)

All problems are from Leon’s book (8th edition).

Section 3.3: 2b, 2c, 2e, 8c, 9d, 14
Section 3.4: 8c, 10, 14c, 15a
MATH 304
Linear Algebra

Lecture 11:
Basis and dimension.
Linear independence

Definition. Let $V$ be a vector space. Vectors $v_1, v_2, \ldots, v_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1v_1 + r_2v_2 + \cdots + r_kv_k = 0,$$

where the coefficients $r_1, \ldots, r_k \in \mathbb{R}$ are not all equal to zero. Otherwise the vectors $v_1, v_2, \ldots, v_k$ are called **linearly independent**. That is, if

$$r_1v_1 + r_2v_2 + \cdots + r_kv_k = 0 \implies r_1 = \cdots = r_k = 0.$$

An infinite set $S \subset V$ is **linearly dependent** if there are some linearly dependent vectors $v_1, \ldots, v_k \in S$. Otherwise $S$ is **linearly independent**.

Remark. If a set $S$ (finite or infinite) is linearly independent then any subset of $S$ is also linearly independent.
Theorem  Vectors $v_1, \ldots, v_k \in V$ are linearly dependent if and only if one of them is a linear combination of the other $k-1$ vectors.

Examples of linear independence.

- Vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$ in $\mathbb{R}^3$.

- Matrices $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

- Polynomials $1, x, x^2, \ldots, x^n, \ldots$
Spanning set

Let $S$ be a subset of a vector space $V$.

**Definition.** The span of the set $S$ is the smallest subspace $W \subset V$ that contains $S$. If $S$ is not empty then $W = \text{Span}(S)$ consists of all linear combinations $r_1v_1 + r_2v_2 + \cdots + r_kv_k$ such that $v_1, \ldots, v_k \in S$ and $r_1, \ldots, r_k \in \mathbb{R}$.

We say that the set $S$ spans the subspace $W$ or that $S$ is a spanning set for $W$.

**Remark.** If $S_1$ is a spanning set for a vector space $V$ and $S_1 \subset S_2 \subset V$, then $S_2$ is also a spanning set for $V$. 
**Basis**

*Definition.* Let $V$ be a vector space. A linearly independent spanning set for $V$ is called a **basis**.

Suppose that a set $S \subset V$ is a basis for $V$.

“Spanning set” means that any vector $v \in V$ can be represented as a linear combination

$$v = r_1v_1 + r_2v_2 + \cdots + r_kv_k,$$

where $v_1, \ldots, v_k$ are distinct vectors from $S$ and $r_1, \ldots, r_k \in \mathbb{R}$. “Linearly independent” implies that the above representation is unique:

$$v = r_1v_1 + r_2v_2 + \cdots + r_kv_k = r'_1v_1 + r'_2v_2 + \cdots + r'_kv_k$$

$$\implies (r_1 - r'_1)v_1 + (r_2 - r'_2)v_2 + \cdots + (r_k - r'_k)v_k = 0$$

$$\implies r_1 - r'_1 = r_2 - r'_2 = \ldots = r_k - r'_k = 0$$
Examples. • Standard basis for $\mathbb{R}^n$: 
$e_1 = (1, 0, 0, \ldots, 0, 0), \ e_2 = (0, 1, 0, \ldots, 0, 0), \ldots, 
 e_n = (0, 0, 0, \ldots, 0, 1).$
Indeed, $(x_1, x_2, \ldots, x_n) = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n.$

• Matrices \[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]
form a basis for $M_{2,2}(\mathbb{R}).$
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

• Polynomials $1, x, x^2, \ldots, x^{n-1}$ form a basis for $P_n = \{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} : a_i \in \mathbb{R}\}.$

• The infinite set $\{1, x, x^2, \ldots, x^n, \ldots\}$ is a basis for $P$, the space of all polynomials.
Let \( \mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n \) and \( r_1, r_2, \ldots, r_k \in \mathbb{R} \). The vector equation \( r_1 \mathbf{v}_1 + r_2 \mathbf{v}_2 + \cdots + r_k \mathbf{v}_k = \mathbf{v} \) is equivalent to the matrix equation \( A\mathbf{x} = \mathbf{v} \), where

\[
A = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k), \quad \mathbf{x} = \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}.
\]

That is, \( A \) is the \( n \times k \) matrix such that vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \) are consecutive columns of \( A \).

- **Vectors** \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) span \( \mathbb{R}^n \) if the row echelon form of \( A \) has no zero rows.

- **Vectors** \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) are linearly independent if the row echelon form of \( A \) has a leading entry in each column (no free variables).
spanning linear independence

no spanning linear independence

spanning no linear independence

no spanning no linear independence
Bases for $\mathbb{R}^n$

Let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbb{R}^n$.

**Theorem 1** If $k < n$ then the vectors $v_1, v_2, \ldots, v_k$ do not span $\mathbb{R}^n$.

**Theorem 2** If $k > n$ then the vectors $v_1, v_2, \ldots, v_k$ are linearly dependent.

**Theorem 3** If $k = n$ then the following conditions are equivalent:

(i) $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$;

(ii) $\{v_1, v_2, \ldots, v_n\}$ is a spanning set for $\mathbb{R}^n$;

(iii) $\{v_1, v_2, \ldots, v_n\}$ is a linearly independent set.
**Example.** Consider vectors $v_1 = (1, -1, 1)$, $v_2 = (1, 0, 0)$, $v_3 = (1, 1, 1)$, and $v_4 = (1, 2, 4)$ in $\mathbb{R}^3$.

Vectors $v_1$ and $v_2$ are linearly independent (as they are not parallel), but they do not span $\mathbb{R}^3$.

Vectors $v_1, v_2, v_3$ are linearly independent since

$$
\begin{vmatrix}
1 & 1 & 1 \\
-1 & 0 & 1 \\
1 & 0 & 1
\end{vmatrix}
= - \begin{vmatrix}
-1 & 1 \\
1 & 1
\end{vmatrix}
= -(-2) = 2 \neq 0.
$$

Therefore $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

Vectors $v_1, v_2, v_3, v_4$ span $\mathbb{R}^3$ (because $v_1, v_2, v_3$ already span $\mathbb{R}^3$), but they are linearly dependent.
Problem. Find a basis for the plane $x + 2z = 0$ in $\mathbb{R}^3$.

The general solution of the equation $x + 2z = 0$ is

$$\begin{cases} 
  x = -2s \\
  y = t \\
  z = s
\end{cases} \quad (t, s \in \mathbb{R})$$

That is, $(x, y, z) = (-2s, t, s) = t(0, 1, 0) + s(-2, 0, 1)$.

Hence the plane is the span of vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$. These vectors are linearly independent as they are not parallel.

Thus $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for the plane $x + 2z = 0$. 
**Theorem 1**  Any vector space has a basis.

**Theorem 2**  If a vector space $V$ has a finite basis, then all bases for $V$ are finite and have the same number of elements.

*Definition.* The **dimension** of a vector space $V$, denoted $\dim V$, is the number of elements in any of its bases.
Examples.  •  \( \dim \mathbb{R}^n = n \)

•  \( \mathcal{M}_{2,2}(\mathbb{R}) \): the space of \( 2 \times 2 \) matrices  
\( \dim \mathcal{M}_{2,2}(\mathbb{R}) = 4 \)

•  \( \mathcal{M}_{m,n}(\mathbb{R}) \): the space of \( m \times n \) matrices  
\( \dim \mathcal{M}_{m,n}(\mathbb{R}) = mn \)

•  \( \mathcal{P}_n \): polynomials of degree less than \( n \)  
\( \dim \mathcal{P}_n = n \)

•  \( \mathcal{P} \): the space of all polynomials  
\( \dim \mathcal{P} = \infty \)

•  \( \{0\} \): the trivial vector space  
\( \dim \{0\} = 0 \)
How to find a basis?

**Theorem** Let $S$ be a subset of a vector space $V$. Then the following conditions are equivalent:

(i) $S$ is a linearly independent spanning set for $V$, i.e., a basis;

(ii) $S$ is a minimal spanning set for $V$;

(iii) $S$ is a maximal linearly independent subset of $V$.

“Minimal spanning set” means “remove any element from this set, and it is no longer a spanning set”.

“Maximal linearly independent subset” means “add any element of $V$ to this set, and it will become linearly dependent”.
**Theorem**  Let $V$ be a vector space. Then

(i) any spanning set for $V$ can be reduced to a minimal spanning set;

(ii) any linearly independent subset of $V$ can be extended to a maximal linearly independent set.

Equivalently, any spanning set contains a basis, while any linearly independent set is contained in a basis.

**Corollary**  A vector space is finite-dimensional if and only if it is spanned by a finite set.
How to find a basis?

Approach 1. Get a spanning set for the vector space, then reduce this set to a basis.

Proposition Let \( v_0, v_1, \ldots, v_k \) be a spanning set for a vector space \( V \). If \( v_0 \) is a linear combination of vectors \( v_1, \ldots, v_k \) then \( v_1, \ldots, v_k \) is also a spanning set for \( V \).

Indeed, if \( v_0 = r_1v_1 + \cdots + r_kv_k \), then

\[
t_0v_0 + t_1v_1 + \cdots + t_kv_k = (t_0r_1 + t_1)v_1 + \cdots + (t_0r_k + t_k)v_k.
\]
How to find a basis?

Approach 2. Build a maximal linearly independent set adding one vector at a time.

If the vector space $V$ is trivial, it has the empty basis.

If $V \neq \{0\}$, pick any vector $v_1 \neq 0$.
If $v_1$ spans $V$, it is a basis. Otherwise pick any vector $v_2 \in V$ that is not in the span of $v_1$.

If $v_1$ and $v_2$ span $V$, they constitute a basis. Otherwise pick any vector $v_3 \in V$ that is not in the span of $v_1$ and $v_2$.

And so on...
Problem. Find a basis for the vector space $V$ spanned by vectors $\mathbf{w}_1 = (1, 1, 0)$, $\mathbf{w}_2 = (0, 1, 1)$, $\mathbf{w}_3 = (2, 3, 1)$, and $\mathbf{w}_4 = (1, 1, 1)$.

To pare this spanning set, we need to find a relation of the form $r_1 \mathbf{w}_1 + r_2 \mathbf{w}_2 + r_3 \mathbf{w}_3 + r_4 \mathbf{w}_4 = \mathbf{0}$, where $r_i \in \mathbb{R}$ are not all equal to zero. Equivalently,

$$
\begin{pmatrix}
1 & 0 & 2 & 1 \\
1 & 1 & 3 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
r_1 \\
r_2 \\
r_3 \\
r_4
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
$$

To solve this system of linear equations for $r_1, r_2, r_3, r_4$, we apply row reduction.
\[
\begin{pmatrix}
1 & 0 & 2 & 1 \\
1 & 1 & 3 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 2 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 2 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(reduced row echelon form)

\[
\begin{cases}
  r_1 + 2r_3 = 0 \\
  r_2 + r_3 = 0 \\
  r_4 = 0
\end{cases}
\iff
\begin{cases}
  r_1 = -2r_3 \\
  r_2 = -r_3 \\
  r_4 = 0
\end{cases}
\]

General solution: \((r_1, r_2, r_3, r_4) = (-2t, -t, t, 0), \ t \in \mathbb{R}\).
Particular solution: \((r_1, r_2, r_3, r_4) = (2, 1, -1, 0)\).
Problem. Find a basis for the vector space \( V \) spanned by vectors \( \mathbf{w}_1 = (1, 1, 0) \), \( \mathbf{w}_2 = (0, 1, 1) \), \( \mathbf{w}_3 = (2, 3, 1) \), and \( \mathbf{w}_4 = (1, 1, 1) \).

We have obtained that \( 2\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3 = 0 \). Hence any of vectors \( \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \) can be dropped. For instance, \( V = \text{Span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4) \).

Let us check whether vectors \( \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4 \) are linearly independent:

\[
\begin{vmatrix}
1 & 0 & 1 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{vmatrix} = \begin{vmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 0
\end{vmatrix} = \begin{vmatrix}
1 & 1 \\
0 & 1
\end{vmatrix} = 1 \neq 0.
\]

They are!!! It follows that \( V = \mathbb{R}^3 \) and \( \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\} \) is a basis for \( V \).
Vectors \( \mathbf{v}_1 = (0, 1, 0) \) and \( \mathbf{v}_2 = (-2, 0, 1) \) are linearly independent.

**Problem.** Extend the set \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) to a basis for \( \mathbb{R}^3 \).

Our task is to find a vector \( \mathbf{v}_3 \) that is not a linear combination of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \).

Then \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) will be a basis for \( \mathbb{R}^3 \).

**Hint 1.** \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) span the plane \( x + 2z = 0 \).

The vector \( \mathbf{v}_3 = (1, 1, 1) \) does not lie in the plane \( x + 2z = 0 \), hence it is not a linear combination of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). Thus \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) is a basis for \( \mathbb{R}^3 \).
Vectors \( \mathbf{v}_1 = (0, 1, 0) \) and \( \mathbf{v}_2 = (-2, 0, 1) \) are linearly independent.

**Problem.** Extend the set \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) to a basis for \( \mathbb{R}^3 \).

Our task is to find a vector \( \mathbf{v}_3 \) that is not a linear combination of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). Then \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) will be a basis for \( \mathbb{R}^3 \).

**Hint 2.** At least one of vectors \( \mathbf{e}_1 = (1, 0, 0) \), \( \mathbf{e}_2 = (0, 1, 0) \), and \( \mathbf{e}_3 = (0, 0, 1) \) is a desired one.

Let us check that \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1 \} \) and \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3 \} \) are two bases for \( \mathbb{R}^3 \):

\[
\begin{vmatrix}
0 & -2 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{vmatrix} = 1 \neq 0, \quad \begin{vmatrix}
0 & -2 & 0 \\
1 & 0 & 0 \\
0 & 1 & 1 \\
\end{vmatrix} = 2 \neq 0.
\]