Programming assignment #3. (Due March 9)

Let $\Omega = (0,1)^2$ be the unit square in $\mathbb{R}^2$. Consider the vector field $\beta = (\alpha,0)$.

Consider the problem:

$$-\nabla^2 u + u + \beta \cdot \nabla u = 1; \quad u|_{\partial \Omega} = 0.$$

Let $N$ be a positive integer. Let $h = \frac{1}{N+1}$ and define the following vertices $x_{ij} = (ih, jh)$ for $0 \leq i, j \leq N + 1$. Denote by $S_{ij}$ the square whose vertices are $x_{ij}$, $x_{i+1,j}$, $x_{i+1,j+1}$, and $x_{i,j+1}$ for $0 \leq i, j \leq N$. Divide $S_{ij}$ into two triangles so that the vertices of these triangles are $x_{ij}$, $x_{i+1,j}$, $x_{i+1,j+1}$, and $x_{ij}$, $x_{i+1,j+1}$, and $x_{i,j+1}$.

Denote by $T_h$ the mesh composed of the triangles above defined. The interior nodes are globally numbered from left to right and bottom to left.

(i) Write a weak formulation for the above problem.

(ii) Write the discrete form of the above problem using $P_1$ finite elements.

(iii) Assemble the matrix and the right-hand side arising from the discrete problem.

(iv) Make sure you assembled the matrix correctly by solving the discrete problem whose source term $f$ corresponds to the following solution $u = x(1-x)y(1-y)$. Use $\alpha = 1$. Plot in log-log scale $\max_{0 \leq i,j \leq N} |u(x_{ij}) - u_h(x_{ij})|$ versus $h$, where $u_h$ denotes the approximate solution. Use $N = 10, 20, 40, 80,$ and $N = 100$.

(v) Solve the problem with $f = 1$ and $N = 50$ using successively $\alpha = 0$, $\alpha = 1$, $\alpha = 10$, $\alpha = 100$, and $\alpha = 1000$. Report what you observe. Explain.